Revised and Updated

Functional Equations

A Problem Solving Approach

Second Edition

Problems from Mathematical Olympiads and other Contests

B. J. Venkatachala

PRISM

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Preface to the second edition

The first edition of this book was well received by the students in the last 10 years. In fact, a student who was in the US, not related to Olympiad, says that the book reads like poetry. What else can an author expect? I have now revised the book and also corrected a few errors that were there in the first edition. I have also included an additional chapter, containing more problems of recent origin. I had to be choosy in my selection, as there are a large number of problems which have appeared in many of the recent National level examinations and IMO's. In my next edition, to add value, I plan to include an appendix on Charles Babbage, who has done incredible work on Functional Equations.

I hope that this second edition will also be received well by the students.

B J Venkatachala
HBCSE, TIFR - Mumbai
April 2013

Preface

The Mathematical Olympiad movement in India has ushered in a new era of mathematical awakening among the people of Indian society. The National Board for Higher Mathematics (NBHM) has risen to the challenge in admirable fashion by steering the Olympiad activity in India and bringing the whole nation under one umbrella of the Indian National Mathematical Olympiad (INMO). The esoteric subject of Mathematics which was hitherto in the confines of a few pedagogue has shred out its shackles and many young raw talents have got attracted to the simplistic beauty of Mathematics. It is really heartening to see that quite a few talented children taking to Mathematics in recent years.

The onus of catering good mathematics to the devouring young minds has naturally fallen upon the mathematical community. As such the needs of the hour are some good books designed exclusively for these needy, brilliant minds. Although some attempts have been made in this direction, the full impact is achievable only by providing our student community with comprehensible basic books. An example to emulate is the erstwhile USSR experiment wherein the truly great top Soviet mathematicians wrote simply beautiful, conceptually clear books and flooded the USSR market with them. As a consequence, the USSR emerged as one of the strongest nations in the World of Mathematics. Many of us owe our interest in mathematics to the problems explored in these books, which were easily available to us when we were students.

It has been felt for a long time that there is a need for a monograph on functional equations to help the aspiring olympiad students. It is one of the few topics never finding its place in a school curriculum. However the analysis of a functional equation does not follow any predetermined the-

ory, but needs only a certain logical thinking and as such is included as one of the topics in olympiads.

It is my desire and sincere wish that this monograph fills in the vacuum which exists in the realm of needy books to the aspiring students. I have written this book in the style of problems and solutions to explain the normal methods which help us in resolving a functional equation. I have collected material from different sources which got accumulated with me for the last several years. I have sincerely tried to acknowledge wherever I could by attributing to the source whenever I was certain of the exactness of the source. I deeply regret and apologise for any of the inadvertent omission in mentioning the source.

I have left a substantial collection of problems as exercises. Some of them are easy and some are really hard. I have not tried to classify them but have provided copious hints to each of the exercises at the end of the book. Those hints should suffice to arrive at a solution. I urge my student readers to sincerely try all those exercises on their own. They may find different solution to each of these exercises and my hints may turn redundant.

I have striven very hard to make this book error free by going through the proofs several times. But still some errors may have crept in. I request the readers to kindly bring any error they may find to my notice either by mail or by e-mail(jana@math.iisc.ernet.in).

This work has benefitted immensely from my discussions with the training faculty of our International Mathematical Olympiad Training Camp and interaction with our olympiad students over the years. I have learnt a lot from the talented bright young minds. I sincerely thank all these students and faculty members.

I have gained a lot from my colleagues C R Pranesachar and C S Yogananda at MO Cell. I am indebted to them in the preparation of this book and I thank them very much

for the encouragement they gave me during this period. I have got full help from C S Yogananda in preparing the manuscript in LaTeX and I am grateful to him.

I also wish to acknowledge: Department of Atomic Energy, National Board for Higher Mathematics, Mumbai, and Department of Mathematics, Indian Institute of Science, Bangalore.

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Dedicated to the memory of
my beloved grandfathers

N Venkappa & B J Venkatagiriyappa

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1

Introduction

We all know what an equation is. We encounter equations quite early in our education. For example, we come across a linear equation while we are studying in lower secondary classes. We are taught how to solve an equation of the form $3x - 2 = 7$. We are told that x is an 'unknown' quantity and we have to solve for x. Without blinking we write down the answer: $x = 3$. When we reach the higher secondary stage, we are exposed to equation of the form, for example, $3x^2 - 6x + 2 = 0$. We are also taught to solve such an equation, called quadratic equation, by several methods. We get two solutions, viz., $x = 1 + (1/\sqrt{3})$ and $x = 1 - (1/\sqrt{3})$. We may need to consider complex numbers, occasionally. As we consider higher degree equations, we begin to feel the complications that arise while solving polynomial equations. But the general theory assures that any polynomial equation in one variable has finitely many solutions.

On the other hand let us consider, for example, the equation

$$5x + 13y = 100,$$

for integers x, y. Specifically, we have to find all pairs (x, y) of integers which satisfy above equation. Perhaps some effort in that direction leads to $(x, y) = (7, 5)$, $(-6, 10)$, $(20, 0)$ and such pairs. It is not hard to see that $(x, y) = (20 - 13k, 5k)$, where k is any integer, is a solution of our equation. Such an equation, called linear Diophantine equation, may possess infinitely many solutions or may not have any solution. We observe that there is only one equation where as we need to determine two unknowns. Another equation of interest is

the 'Pythagoras equation': solve the equation

$$x^2 + y^2 = z^2$$

for integers x, y, z. Thus we have to determine three unknowns. The general solution is given by

$$x = k(m^2 - n^2), y = 2kmn, z = k(m^2 + n^2),$$

where k, m, n are natural numbers.

An equation in which unknowns are functions is called a *functional equation.* We are asked to find all functions satisfying some given relation (relations). As an example consider the question: find all functions f satisfying the equation $f(-x) = -f(x)$. An immediate doubt that crosses our mind is: where is f defined and what are the values it takes? Thus the above problem is not *well posed.* We must specify the domain and the range of f before seeking any answer to the question. If we modify our problem to: find all $f : \mathbb{R} \to \mathbb{R}$ such that $f(-x) = -f(x)$, then the problem makes sense. Any solution of this problem is called an *odd* function on the real line.

Unfortunately, even this problem is a 'goose chase' in a 'wild forest'. We can construct any number of solutions to this problem. For example $f(x) = x^{2n+1}$ for any natural number n is a solution. Or $f(x) = \sin x$ is a solution. We can define f arbitrarily on nonnegative real numbers and then set $f(x) = -f(-x)$ for negative x. Thus a functional equation may possess a large number of solutions. To narrow down the number of solutions, we may need to impose additional conditions on the nature of f in terms of either equations or properties of the function. Suppose for example, we require that f should also satisfy the equation $f(xy) = x^2 f(y)$ for all reals x, y. We obtain

$$\begin{aligned} -f(xy) = f(-xy) &= f((-x)y) \\ &= (-x)^2 f(y) = x^2 f(y) = f(xy). \end{aligned}$$

It follows that $f(xy) = 0$ for all reals x, y. Taking $y = 1$, we see that $f(x) = 0$ for all x in $\mathbb{R}$. Thus the equations

$$f(-x) = -f(x), \quad f(xy) = x^2 f(y),$$

has only one solution: $f(x) = 0$ for all real x.

Here we observe that a single equation can lead to multitude of solutions, where as just an additional equation or condition may drastically reduce the number of solutions. It should be emphasized that the number of equations is not related to the number of solutions as in the case of linear equations. We shall also see later how a single equation(or the same system of equations) can hide information about seemingly unrelated functions. This inherent capacity of a functional equation for containing lot of information about unrelated functions make it more intractable than the class of other types of equations. And the beauty of a functional equation also lies in its strength to hold information about distinct classes of functions.

While solving a functional equation, we need to keep in mind the property of domain of the functions, their range and also the given conditions on the functions. We shall see that various well known sets with nice structures form the domain and range of functions: we use $\mathbb{N}$, the set of all natural numbers; $\mathbb{Z}$, the set of all integers; $\mathbb{Q}$, the set of all rational numbers; and $\mathbb{R}$ the set of all real numbers. Occasionally, we may need $\mathbb{C}$, the set of all complex numbers and $\mathbb{R}^n$, the Euclidean space of dimension n. We may also use $\mathbb{N}_0$, the set of all nonnegative integers; $\mathbb{Q}_0$, the set of all nonnegative rational numbers; $\mathbb{Q}^+$, the set of all positive rational numbers; $\mathbb{R}_0$, the set of all nonnegative real numbers; and $\mathbb{R}^+$, the set of all positive real numbers. We shall also use a variety of conditions on the functions like monotonicity, boundedness, continuity etc., which would help us in fixing the solutions of functional equations.

The study of functional equations has a long history and

is associated with giants like D'Alembert, Euler, Cauchy, Gauss, Legendre, Darboux, Abel and Hilbert. D'Alembert arrived at the problem of solving the equation

$$f(x+y)+f(x-y)=g(x)h(y)$$

for functions f, g, h on $\mathbb{R}$ in his work on vibrating strings. Cauchy investigated equations of the form

$$f(x+y)=f(x)+f(y), f(x+y)=f(x)f(y),$$

$$f(xy)=f(x)+f(y), f(xy)=f(x)f(y),$$

which made their appearances in the problems of measuring *Areas* and *Normal Probability Distribution.* Thus the study of functional equations arose from practical considerations. The areas of Differential equations, Integral equations and Difference equations which are very useful in solving many practical problems also fall in to the category of functional equations. However we do not consider them here and concentrate only on pure functional equations.

In the ensuing chapters, we shall see how different methods can be employed for solving functional equations. The special structural properties of domain, range and also the condition(s) on the functions which are sought will play a pivotal role in the method of solving a functional equation. Different equations need different approaches and different perspective. These aspects are emphasized in the next few chapters while solving functional equations. We consider equations on $\mathbb{N}$ and those equations posed on $\mathbb{Z}$ separately in chapter 2. Equations on $\mathbb{Q}$ and $\mathbb{R}$ are considered in chapter 3, but without having any further hypothesis on the functions. Cauchy's equation(s) and those equations which specially use Cauchy's equation(s) are treated separately in chapter 4. In chapter 5, we see how additional conditions on functions could be used to solve functional equations, although we do not get the most general solution in such cases. Each

of these chapter ends with a substantial number of exercises, some easy and some hard. These problems are provided with adequate hints at the end. Some of the problems may possess many different solutions. It is extremely instructive and exhilarating to construct new solutions to the given problems.

It is my experience over the years that use of elementary ideas while solving the given functional equation will go a long way in revealing the structure of that equation and natural additional conditions to be imposed would manifest on their own. It is advisable to pursue the equation till there is no further go before looking for extra condition that has to be put on the function either as a property or as another equation. These ideas are made clear in the next few chapters.

2

Equations On Natural Numbers

The functional equations are generally posed on sets with some nice structures. We know many such sets which we indispensably come across in our mathematical problems. The most important among them are $\mathbb{N}$, the set of all natural numbers; $\mathbb{Z}$, the set of all integers; $\mathbb{Q}$, the set of all rational numbers; and $\mathbb{R}$, the set of all real numbers. While dealing with polynomial equations, we also need $\mathbb{C}$, the set of all complex numbers. All these number systems are extremely important while solving problems. These are also concrete realizations of many abstract things which evolve through the Mathematician's ever active mind and helps him to consolidate his ideas.

The set $\mathbb{N}$ of all natural numbers is the first number system we normally encounter. There are several important things we learn about $\mathbb{N}$. There are natural concepts of addition and multiplication on $\mathbb{N}$. (Multiplication is in some sense repeated addition here.) Moreover we also have here an inherent way of comparing two numbers: we know the meaning of saying a number is smaller or bigger than another number. Thus $\mathbb{N}$ is equipped with addition, multiplication and ordering. While dealing with functional equations on $\mathbb{N}$, these properties play a fundamental role. In some problems, it may be sufficient to use one of these properties. But some problems need full force of all these fundamental properties. We illustrate these ideas with several problems.

Problem 2.1 Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that

(a) $f(2) = 2$;

(b) $f(mn) = f(m)f(n)$ for all m, n in $\mathbb{N}$;

(c) $f(m) < f(n)$ whenever $m < n$.

(The property (b) is often referred to as *multiplicativity* condition. The condition (c) simply says that f is strictly increasing on $\mathbb{N}$.)

Solution: One important tool we have while solving equations on $\mathbb{N}$ is the *Principle of Mathematical Induction.* This simple looking principle gives us a very powerful method often extending to unexpected places. We use it here to solve our problem.

We see from (b) that

$$f(1) = f(1 \cdot 1) = f(1)^2.$$

Since we are in $\mathbb{N}$ (remember $\mathbb{N}$ does not contain 0), we conclude that $f(1) = 1$. Similarly, taking $m = n = 2$ in (b), we obtain $f(4) = f(2)^2 = 4$. Now using (c) we can fix $f(3)$. Because $2 < 3 < 4$, we know from (c) that $f(2) < f(3) < f(4)$. But $f(2) = 2$ and $f(4) = 4$, and 3 is the only natural number between 2 and 4. We conclude that $f(3) = 3$, thus getting the value of $f(3)$.

Now we use this information to conclude that $f(6) = f(2 \cdot 3) = f(2)f(3) = 6$. We use the fact that $4 < 5 < 6$ and (c) to conclude that $4 = f(4) < f(5) < f(6) = 6$. This fixes the value of $f(5)$, namely, $f(5) = 5$. We now know how to proceed in order to complete the proof by induction. Suppose we have proved that $f(1) = 1, f(2) = 2, \ldots, f(2k) = 2k$, for some natural number k. Using (b), we have

$$f(2k + 2) = f(2)f(k + 1) = 2(k + 1) = 2k + 2.$$

We have used the fact that $k + 1 \leq 2k$ and the induction hypothesis. Since (c) implies that $2k = f(2k) < f(2k+1) < f(2k + 2) = 2k + 2$, we conclude $f(2k + 1) = 2k + 1$.

It follows by principle of induction that $f(n) = n$ for all natural numbers. ■

We have used two important properties in the above solution: the ordering on natural numbers and the principle of induction. We have effectively used all the three conditions we are given. Let us see what happens if we relax the conditions on f.

Problem 2.2 Find all functions $f : \mathbb{N} \to \mathbb{N}$ which satisfy

(a) $f(2) = 2$;

(b) $f(mn) = f(m)f(n)$ for all m, n in $\mathbb{N}$ satisfying the condition $\gcd(m, n) = 1$;

(c) $f(m) < f(n)$ whenever $m < n$.

Solution: We show as in problem 1 that $f(n) = n$ for all natural numbers. Using the same method, we can prove that $f(1) = 1$. But unfortunately we cannot conclude, as we have done there, that $f(4) = 4$. The condition (b) in problem 1 is applicable to all pairs $\{m, n\}$ of natural numbers, where as the condition (b) in the present problem is applicable only to relatively prime pair $\{m, n\}$ of natural numbers. Thus (b) cannot be invoked to conclude $f(4) = f(2)^2$. Some how if we can manage to prove that $f(3) = 3$, then the method of solution in problem 1 can be effectively used to get the desired conclusion. For example we can get $f(6) = f(2 \cdot 3) = f(2)f(3) = 6$. Now we use (c) to infer that $3 = f(3) < f(4) < f(5) < f(6) = 6$ and this immediately tells us that $f(4) = 4$ and $f(5) = 5$. We can complete the proof by induction by using, for example, $\gcd(k, k-1) = 1$ for any natural number $k \geq 2$. Suppose we have proved that $f(k) = k$ for all $k \leq n$. Then we get from (b) the information $f((n-1)n) = f(n-1)f(n) = (n-1)n$. Invoking (c), we

obtain

$$n = f(n) < f(n+1) < f(n+2) < \cdots < f(n^2 - n - 1) < f(n^2 - n) = f((n-1)n) = (n-1)n.$$

This leads to

$$\begin{aligned} f(n+1) &= n+1, \quad f(n+2) = n+2, \ldots, \\ f\big((n-1)n-1\big) &= (n-1)n-1, \\ f\big((n-1)n\big) &= (n-1)n. \end{aligned}$$

This completes the proof of our claim by the principle of mathematical induction.

Thus the completion of proof hinges on getting the key result that $f(3) = 3$. We proceed to prove this as follows. We have

$$f(3)f(5) = f(3 \cdot 5) = f(15) < f(18) = f(2 \cdot 9) = f(2)f(9) = 2f(9),$$

and

$$f(9) < f(10) = f(2 \cdot 5) = f(2)f(5) = 2f(5).$$

We have repeatedly used (b) and (c). These two inequalities show that

$$f(3)f(5) < 2f(9) < 4f(5),$$

giving $f(3) < 4$. Since $2 = f(2) < f(3) < 4$, we conclude that $f(3) = 3$. This proves our claim and hence completes the solution of the problem. ■

The value of $f(2)$ cannot be arbitrarily given. If we take $f(2) = k$, then there may not exist a function satisfying the conditions (b) and (c) of problem 2 for a particular value of k, as the following problem shows.

Problem 2.3 Show that there does not exist a function $f : \mathbb{N} \to \mathbb{N}$ which satisfy

(a) $f(2) = 3$;

(b) $f(mn) = f(m)f(n)$ for all m, n in $\mathbb{N}$;

(c) $f(m) < f(n)$ whenever $m < n$.

Solution: Suppose the contrary. Let $f : \mathbb{N} \to \mathbb{N}$ be a function satisfying (b) and (c) such that $f(2) = 3$. Let us write $f(3) = l$. Using the inequality $2^3 < 3^2$, and (b) and (c), we obtain

$$3^3 = f(2)^3 = f(2^3) < f(3^2) = f(3)^2 = l^2.$$

This shows that $l > 5$. Similarly using $3^3 < 2^5$, and (b) and (c), we get

$$l^3 = f(3)^3 = f(3^3) < f(2^5) = 3^5 = 243 < 343 = 7^3.$$

This implies that $l < 7$. We conclude that $l = 6$; i.e., $f(3) = 6$. However we also know that $3^8 = 6561 < 8192 = 2^{13}$. Again using (b) and (c), we infer

$$6^8 = f(3^8) < f(2^{13}) = 3^{13}.$$

This simplifies to $2^8 < 3^5$. But $2^8 = 256$ where as $3^5 = 243$. Thus we arrive at an absurd conclusion that $256 < 243$. This contradiction proves that there is no function of the desired type. ■

On the other hand if we impose $f(2) = 4$, then $f(n) = n^2$ satisfies all the conditions.

If we relax the condition (c), what happens? Note the condition (c) simply says that f is strictly increasing.

Problem 2.4 Show that there are infinitely many functions $f : \mathbb{N} \to \mathbb{N}$ such that

(a) $f(2) = 2$;

(b) $f(mn) = f(m)f(n)$ for all m, n in $\mathbb{N}$.

Solution: Here we use another important property of natural numbers: given any natural number n, there exists a unique set $\{p_1, p_2, \ldots, p_k\}$ of primes and a unique set $\{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ of positive integers such that

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}.$$

This is called the *prime decomposition* of n. The condition (b) shows that in order to know f, it is sufficient to know its values at each prime. We have its value at $p = 2$. We can define f arbitrarily on primes $\neq 2$ and use (b) to extend it to all other natural numbers using prime decomposition. Note that (b) forces $f(1) = 1$.

For example let us enumerate the set of all prime numbers as an increasing sequence: $3 = p_2 < p_3 < p_4 < \cdots$. Define for each $k \geq 1$, the function f_k by $f_k(p_j) = p_{j+k}$ for all $j \geq 2$. If $n = q_1^{\alpha_1} q_2^{\alpha_2} \cdots q_l^{\alpha_l}$ is the prime decomposition of n, then we define $f_k(n) = f_k(q_1)^{\alpha_1} f_k(q_2)^{\alpha_2} \cdots f_k(q_l)^{\alpha_l}$. Thus $f_1(3) = 5$, $f_1(5) = 7$, $f_1(7) = 11$ etc. If we want $f_1(15)$, we can get it as $f_1(15) = f_1(3)f_1(5) = 35$. Similarly $f_1(16) = f_1(2)^4 = 16$. Observe that $f_1(16) < f_1(15)$. ∎

In the above problems, we considered those functions whose range is also a subset of the set of all natural numbers. This helped us in fixing the values of f. Specifically we have used the fact that there is no natural number between n and $n + 1$. But such a thing is not possible if we use either $\mathbb{Q}$, the set of all rational numbers or $\mathbb{R}$, the set of all real numbers. In these cases, we can squeeze in another number

between any two given numbers: there is always a rational number between any two rationals and a real number (in fact a rational number can also be found) between any two real numbers. Suppose we take the range of f to be a subset of $\mathbb{R}$. What can we say about the possible solutions of problem 1?

Problem 2.5 Let $f : \mathbb{N} \to [1, \infty)$ such that

(a) $f(2) = 2$;

(b) $f(mn) = f(m)f(n)$ for all m, n in $\mathbb{N}$;

(c) $f(m) < f(n)$ whenever $m < n$.

Prove that $f(n) = n$ for all natural numbers n.

Solution: We proceed as in earlier problem to get $f(1) = 1$ and $f(4) = 4$. Unfortunately, we cannot use (c) to fix $f(3)$. Although we have

$$2 = f(2) < f(3) < f(4) = 4,$$

we cannot conclude that $f(3) = 3$ as we have done in problem 1. Since the range is a subset of real numbers, there are infinitely many(in fact uncountably many) real numbers which are possible candidates for $f(3)$. Thus there is a need to adopt a different strategy to get our answer.

Using (b) and induction, we can prove that $f(2^k) = 2^k$ for every natural number k. Let us take any $m \in \mathbb{N}$ and suppose $f(m) = l$. Then $f(m^n) = l^n$ for all $n \in \mathbb{N}$. If k is such that $2^k \leq m^n < 2^{k+1}$, then using (b) and (c) we obtain

$$2^k \leq l^n < 2^{k+1}.$$

Thus we get the inequality

$$\frac{1}{2} < \left(\frac{m}{l}\right)^n < 2, \tag{1}$$

valid for all natural numbers n. If $m > l$, choose n such that $n > l/(m-l)$. Then we have

$$\left(\frac{m}{l}\right)^n = \left(1 + \frac{m-l}{l}\right)^n > 1 + n\left(\frac{m-l}{l}\right) > 2,$$

by our choice of n. This contradicts the right hand side inequality of (1). If $m < l$, we choose $n > m/(l-m)$. In this case we get

$$\left(\frac{l}{m}\right)^n = \left(1 + \frac{l-m}{m}\right)^n > 1 + n\left(\frac{l-m}{m}\right) > 2,$$

again by our choice of n. We thus obtain $(m/l)^n < 1/2$. But this contradicts the left hand side inequality of (1).

We conclude that $l = m$ thus forcing $f(m) = m$ for all natural numbers m. ■

There is yet another useful property of natural numbers which is often used in solving functional equations on $\mathbb{N}$. This is called the *Well Ordering Principle* on $\mathbb{N}$. It asserts that any nonempty subset of $\mathbb{N}$ has the least element. Thus if $S \subset \mathbb{N}$ and if $S \neq \emptyset$, then there exist a unique $m \in S$ such that $m \leq n$ for all $n \in S$. This simple looking, intuitively clear principle is extremely powerful. It is in fact equivalent to the Principle of Mathematical Induction.(Prove this!) We shall see how this form of Mathematical Induction can be used to solve functional equations on $\mathbb{N}$.

Problem 2.6 If $f : \mathbb{N} \to \mathbb{N}$ is such that

$$f(n+1) > f(f(n)), \quad \text{for all natural numbers } n,$$

prove that $f(n) = n$, for all $n \in \mathbb{N}$. (IMO-1977)

Solution: Let d be the least element of the range of f; i.e.,

$$d = \min\{f(n) \,:\, n \in \mathbb{N}\}.$$

By the well ordering principle afore mentioned, such an element d exists and it is unique. Let $m \in \mathbb{N}$ be such that $d = f(m)$. If $m > 1$, then we see that $d = f(m) > f(f(m-1))$. Thus we get a new element $f(m-1)$ whose f-value is smaller than d. But this contradicts the choice of d as the least element of $\{f(n) : n \in \mathbb{N}\}$. We conclude that $m = 1$. Thus f has a unique minimum at 1.

Now consider the set $\{f(n) : n \geq 2\}$. We can infer, as in the previous paragraph, that this set has the least element and this least value is $f(2)$. Moreover $f(1) < f(2)$, by the choice of $f(1)$. In fact $f(1) = f(2)$ forces $f(1) > f(f(1))$ contradicting the choice of $f(1)$. This can be continued to get

$$f(1) < f(2) < f(3) < \cdots < f(n) < \cdots . \tag{1}$$

Note that $f(1) \geq 1$. This bound along with (1) shows that $f(k) \geq k$ for all natural numbers k. Suppose $f(k) > k$ for some k. Then $f(k) \geq k + 1$. Using (1) we conclude that $f(k+1) \leq f(f(k))$. But this contradicts the given condition: $f(k+1) > f(f(k))$. We conclude that $f(k) = k$ for all natural numbers k.

Alternately, as in the previous step, we show that $f(1)$ is the least element of the set $\{f(n) : n \in \mathbb{N}\}$ and $f(2)$ is the least element of the set $\{f(n) : n \geq 2\}$. If $f(1) > 1$, then we must have $f(1) \geq 2$ and hence $f(f(1)) \geq f(2)$ by the least property of $f(2)$. But this contradicts the given relation. Hence $f(1) = 1$. Now consider $g(n) = f(n+1) - 1$. We see that

$$\begin{aligned} g\big(g(n)\big) &= g\big(f(n+1) - 1\big) \\ &= f\big(f(n+1)\big) - 1 < f(n+2) - 1 = g(n+1). \end{aligned}$$

Thus g satisfies the same relation as that satisfied by f. It follows that $g(1) = 1$ and hence $f(2) = 2$. By induction, we prove that $f(n) = n$ for all n. ∎

In many problems, we exploit the property that the set of all natural numbers is precisely the set of positive integers from among all integers. Thus we use the fact that $m \geq 1$ for all natural numbers m. In other words, 1 is the least element of $\mathbb{N}$.

Problem 2.7 Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that

$$f\big(f(m) + f(n)\big) = m + n, \tag{1}$$

for all $m, n \in \mathbb{N}$.

Solution: We show that f is the identity function on $\mathbb{N}$; i.e., $f(n) = n$, for all $n \in \mathbb{N}$. We observe that (1) forces f to be an one-one function. In fact, we see that

$$\begin{aligned} f(m) = f(n) &\Rightarrow f(m) + f(n) = f(n) + f(n) \\ &\Rightarrow f\big(f(m) + f(n)\big) = f\big(f(n) + f(n)\big) \\ &\Rightarrow m + n = n + n \\ &\Rightarrow m = n. \end{aligned}$$

If $k < n$, then we have from (1),

$$\begin{aligned} f\big(f(m+k) + f(n-k)\big) &= (m+k) + (n-k) \\ &= m + n = f\big(f(m) + f(n)\big). \end{aligned}$$

Using the fact that f is an one-one function, this implies

$$f(m+k)+f(n-k) = f(m)+f(n), \ \forall m, n, k \in \mathbb{N}, k < n. \tag{2}$$

Suppose, if possible, $f(1) = b > 1$. Then b is at least 2 so that $b \geq 2$. Moreover we get

$$f(2b) = f\big(f(1) + f(1)\big) = 2,$$

and

$$f(b+2) = f\big(f(1) + f(2b)\big) = 1 + 2b.$$

If $b = 2$, then the above relations lead to $f(4) = 2$ and $f(4) = 5$, which is absurd. Hence b is necessarily larger than 2. Now using (2), we obtain,

$$f(2b)+f(1) = f\big(2b-(b-2)\big)+f(1+b-2) = f(b+2)+f(b-1).$$

This leads to $2+b = 1+2b+f(b-1)$, giving us $f(b-1) = 1-b$. But this is impossible since $1-b < 0$ and $f(b-1) \geq 1$. We conclude that $b = 1$. Thus we have

$$2 = f(2b) = f(2).$$

Now we can complete the proof by induction. Suppose $f(k) = k$ for all $k \leq n$. Then using the given equation we obtain

$$n+1 = f\big(f(n) + f(1)\big) = f(n+1),$$

since $f(n) = n$ by induction hypothesis. It follows that $f(n) = n$ for all $n \in \mathbb{N}$. ∎

Problem 2.8 Let $f : \mathbb{N} \to \mathbb{N}$ be a strictly increasing function such that $f\big(f(n)\big) = 3n$, for all natural numbers n. Find $f(2001)$.

Solution: An immediate consequence of the given relation is that f is an one-one function on $\mathbb{N}$. We have from the given relation,

$$f(3n) = f\Big(f\big(f(n)\big)\Big) = 3f(n), \quad \forall n \in \mathbb{N}.$$

It follows that $f(3) = 3f(1)$. If $f(1) = 1$, then we obtain

$$3 = 3 \cdot 1 = f\big(f(1)\big) = f(1) = 1,$$

which we is absurd. It follows that $f(1) > 1$ and hence

$$3 = f\big(f(1)\big) > f(1) > 1,$$

where we have used the fact that f is strictly increasing. The only possibility, therefore, is $f(1) = 2$. This in turn implies that $f(2) = f\big(f(1)\big) = 3$. Since $2001 = 3 \cdot 667$, it is sufficient to compute $f(667)$.

We shall get an expression for $f(k)$, for any k in $\mathbb{N}$. We observe that

$$f(3) = 3f(1) = 6, \quad f(6) = f(3 \cdot 2) = 3f(2) = 9.$$

Since f is strictly increasing, we also note that,

$$6 = f(3) < f(4) < f(5) < f(6) = 9.$$

This completely determines the intermediate values of f; $f(4) = 7$, $f(5) = 8$. These values in turn give, $f(7) = f\big(f(4)\big) = 3 \cdot 4 = 12$, $f(8) = f\big(f(5)\big) = 15$, $f(9) = f\big(f(6)\big) = 18$. Now using $f(7) = 12$, we obtain $f(12) = f\big(f(7)\big) = 3 \cdot 7 = 21$. The values $f(9) = 18$ and $f(12) = 21$ together with the fact that f is strictly increasing now determines $f(10) = 19$ and $f(11) = 20$.

Suppose for some k, we have $f(k) = n$ and $f(k+1) = n+1$. Then we see that

$$f(n) = f\big(f(k)\big) = 3k, \quad f(n+1) = f\big(f(k+1)\big) = 3k+3.$$

If $f(k) = n$ and $f(k+1) = n+3$, then

$$f(n) = 3k, \quad f(n+3) = 3k+3,$$

and these values fix $f(n+1)$ and $f(n+2)$;

$$f(n+1) = 3k+1, \quad f(n+2) = 3k+2.$$

Let n be such that $3^m \leq n < 2 \cdot 3^m$, for some m. In this case

$$f(3^m) = 3^m f(1) = 2 \cdot 3^m,$$

$$f(2 \cdot 3^m) = f\big(f(3^m)\big) = 3 \cdot 3^m = 3^{m+1}.$$

Note that, because f is strictly increasing,

$$2 \cdot 3^m = f(3^m) < f(3^m + 1) < \cdots < f(3^m + 3^m - 1) < f(2 \cdot 3^m) = 3^{m+1},$$

and hence we get

$$f(3^m + j) = 2 \cdot 3^m + j, \quad \text{for } 0 \leq j \leq 3^m.$$

Thus $f(n) = n + 3^m$ for all n such that $3^m \leq n \leq 2 \cdot 3^m$.

If $2 \cdot 3^m \leq n \leq 3^{m+1}$, then $n = 2 \cdot 3^m + j$, where $0 \leq j \leq 3^m$. Hence

$$\begin{aligned} f(n) = f(2 \cdot 3^m + j) &= f\big(f(3^m + j)\big) \\ &= 3(3^m + j) \\ &= 3^{m+1} + 3j = 3n - 3^{m+1}. \end{aligned}$$

We have thus obtained the following description of $f(n)$:

$$f(n) = \begin{cases} n + 3^m, & \text{for } 3^m \leq n \leq 2 \cdot 3^m, \\ 3n - 3^{m+1}, & \text{for } 2 \cdot 3^m \leq n \leq 3^{m+1}. \end{cases}$$

Since $2001 = 3 \cdot 667$, we obtain

$$f(2001) = 3f(667).$$

We observe that $3^5 = 243$, $2 \cdot 3^5 = 486$ and $3^6 = 729$. Thus 667 lies between $2 \cdot 3^5$ and 3^6. Using the description of f, we obtain

$$f(667) = 3 \cdot 667 - 3^6 = 3(667 - 243) = 1272.$$

Thus the required value is $f(2001) = 3(1272) = 3816$. ■

Problem 2.9 Find all functions $f : \mathbb{N}_0 \to \mathbb{N}_0$, satisfying the equation

$$f\big(f(n)\big) + f(n) = 2n + 3k, \quad \text{for all } n \in \mathbb{N}_0,$$

where k is a fixed natural number.

Solution: Here we illustrate how to use some of the known facts about difference equations to solve some functional equations. Let us put $f(0) = m$. We develop a method to compute m.

We have

$$f(m) = f\big(f(0)\big) = 3k - f(0) = 3k - m;$$

$$f(3k - m) = f\big(f(m)\big) = 2m + 3k - f(m) = 3m;$$

$$\begin{aligned} f(3m) &= f\big(f(3k - m)\big) \\ &= 2(3k - m) + 3k - f(3k - m) = 9k - 5m. \end{aligned}$$

These relations show that f takes a combination of m and k to another combination of m and k. We are thus led to look for a recurrence relation satisfied by the coefficients of m and n. Define two sequences $\langle u_n \rangle$ and $\langle v_n \rangle$ by

$$u_1 = 1, \quad u_2 = -1, \quad u_{n+1} = -u_n + 2u_{n-1}, \text{ for } n \geq 2;$$

$$v_1 = 0, \quad v_2 = 3, \quad v_{n+1} = -v_n + 2v_{n-1} + 3, \text{ for } n \geq 2.$$

Suppose for some n, we have $f(u_{n-1}m + v_{n-1}k) = u_n m + v_n k$. Then we have

$$\begin{aligned} f(u_n m + v_n k) &= f\big(f(u_{n-1}m + v_{n-1}k)\big) \\ &= 2(u_{n-1}m + v_{n-1}k) + 3k \\ &\qquad - f(u_{n-1}m + v_{n-1}k) \\ &= 2u_{n-1}m + (2v_{n-1} + 3)k - (u_n m + v_n k) \\ &= (-u_n + 2u_{n-1})m + (-v_n + 2v_{n-1} + 3)k \\ &= u_{n+1}m + v_{n+1}k. \end{aligned}$$

Since this is true for $n = 1$, we conclude that

$$f(u_n m + v_n k) = u_{n+1} m + v_{n+1} k, \text{ for all } n \in \mathbb{N}.$$

Thus we have to solve two difference equations:

$$u_{n+1} + u_n - 2u_{n-1} = 0, \quad u_1 = 1, \quad u_2 = -1;$$

$$v_{n+1} + v_n - 2v_{n-1} = 3, \quad v_1 = 0, \quad v_2 = 3.$$

These two have the same homogeneous equation, and the auxiliary equation is $x^2 + x - 2 = 0$ which has roots $x = 1$ and $x = -2$. The theory of linear difference equations tells us that the general form of u_n and v_n are given by

$$u_n = A + B(-2)^n, \quad v_n = C + D(-2)^n + n,$$

where the constants A, B, C, D have to be determined by the initial conditions $u_1 = 1, u_2 = -1, v_1 = 0, v_2 = 3$. Note that the non-homogeneous part of the equation for v_n is taken care by the particular solution $v_n = n$ of the second equation. Using the initial conditions, we determine

$$A = \frac{1}{3}, \quad B = -\frac{1}{3}, \quad C = -\frac{1}{3}, \quad D = \frac{1}{3}.$$

Thus the general solutions are given by

$$u_n = \frac{1}{3}\Big(1 + (-1)^{n+1} 2^n\Big),$$

and

$$v_n = -\frac{1}{3}\Big(1 + (-1)^{n+1} 2^n\Big) + n = -u_n + n.$$

Since the range of f is nonnegative integers, we have $f(l) \geq 0$ for all $l \in \mathbb{N}_0$. This implies that

$$u_n(m - k) + nk = u_n m + v_n k = f(u_{n-1} m + v_{n-1} k) \geq 0,$$

for all $n \geq 2$. However the form of u_n tells us that $u_n > 0$ for all odd values of n and $u_n < 0$ for all even values of n. It follows that

$$m - k + \frac{nk}{u_n} \geq 0, \quad \text{if } n \text{ is odd},$$

and

$$m - k + \frac{nk}{u_n} \leq 0, \quad \text{if } n \text{ is even}.$$

But $(nk/u_n) \to 0$ as $n \to \infty$. Using the above behavior, we conclude that $m - k \geq 0$ as well as $m - k \leq 0$. It follows that $m = k$ and thus $f(0) = k$.

Define a new function $g : \mathbb{N}_0 \to \mathbb{N}_0$ by setting

$$g(n) = f(n+1) - 1.$$

We observe that

$$\begin{aligned} g\big(g(n)\big) &= f\big(g(n)+1\big) - 1 \\ &= f\big(f(n+1)\big) - 1 \\ &= 2(n+1) + 3k - f(n+1) - 1 \\ &= 2n + 3k - g(n). \end{aligned}$$

Thus g satisfies the functional equation

$$g\big(g(n)\big) + g(n) = 2n + 3k,$$

which is same as the original equation. Now the same analysis shows that $g(0) = k$. Since $g(0) = f(1) - 1$, we conclude that $f(1) = 1 + k$.

Suppose we have proved that $f(j) = j + k$ for all $j < m$. Then this implies that $g(m-1) = (m-1)+k$ since g satisfies the same equation as the one by f and hence

$$f(m) = g(m-1) + 1 = (m-1) + k + 1 = m + k.$$

It follows by the principle of induction that

$$f(n) = n + k, \quad \text{for all } n \in \mathbb{N}_0.$$

Alternately, for a fixed m, we define $b_n = f^n(m)$, where $f^n(m) = f\big(f^{n-1}(m)\big)$. Thus the given relation implies that $b_2 + b_1 = 2m + 3k$. We also get from the equation the recurrence relation $b_{n+2} + b_{n+1} = 2b_n + 3k$. Its general solution, as in the earlier discussion, is

$$b_n = C + D(-2)^n + nk.$$

Using the fact that $b_n \geq 0$ for all n, it follows that $D = 0$. Hence $b_n = C + nk$, for some constant C. Thus we obtain $2m + 3k = b_2 + b_1 = 2C + 3k$ giving $C = m$. This leads to $f(m) = b_1 = C + k = m + k$. This can be carried out for any given m. We conclude that $f(m) = m + k$, for all $m \in \mathbb{N}_0$. ■

Problem 2.10 The function f is defined on the set of all positive integers as follows:

$$\begin{aligned} &f(1) = 1, \quad f(3) = 3, \quad f(2n) = f(n), \\ &f(4n+1) = 2f(2n+1) - f(n), \\ &f(4n+3) = 3f(2n+1) - 2f(n). \end{aligned}$$

Find the number of n with $f(n) = n$, $1 \leq n \leq 1988$.

(IMO 1988)

Solution: This problem is one of the strange problems that surfaced among functional equations. At the same time, it is also one of the most elegant problems ever proposed in IMO's. The computation of the first few values may give

some hint to the solution of the problem.

$$\begin{aligned}
f(1) &= 1, f(3) = 3, f(2) = f(1) = 1, \\
f(4) &= f(2) = 1, \\
f(5) &= 2f(3) - f(1) = 6 - 1 = 5, \\
f(6) &= f(3) = 3, \\
f(7) &= 3f(3) - 2f(1) = 9 - 2 = 7, \\
f(8) &= f(4) = 1, \\
f(9) &= 2f(5) - f(2) = 10 - 1 = 9, \\
f(10) &= f(5) = 5, \\
f(11) &= 3f(5) - 2f(2) = 15 - 2 = 13, \\
f(12) &= f(6) = 3, \\
f(13) &= 2f(7) - f(3) = 14 - 3 = 11, \\
f(14) &= f(7) = 7, \\
f(15) &= 3f(7) - 2f(3) = 21 - 6 = 15, \\
f(16) &= f(8) = 1.
\end{aligned}$$

We write these in binary notation, i.e., we express first few relations above in base 2.

$$\begin{aligned}
f((1)_2) &= (1)_2, \quad f((10)_2) = f(2) = 1 = (01)_2, \\
f((11)_2) &= f(3) = 3 = (11)_2, \\
f((100)_2) &= f(4) = 1 = (001)_2, \\
f((101)_2) &= f(5) = 5 = (101)_2, \\
f((110)_2) &= f(6) = 3 = (011)_2, \\
f((111)_2) &= f(7) = 7 = (111)_2, \\
f((1000)_2) &= f(8) = 1 = (0001)_2.
\end{aligned}$$

The pattern is clear by this time. We guess that the value of $f(n)$ is obtained by first writing n in base 2 , then reversing the digits and converting back this binary string to the

decimal representation. Thus our claim is: if

$$n = a_k 2^k + a_{k-1} 2^{k-1} + \cdots + a_0,$$

where $a_j \in \{0, 1\}$ for $0 \leq j \leq k$, then $f(n)$ is given by

$$f(n) = a_0 2^k + a_1 2^{k-1} + \cdots + a_k.$$

We prove this claim by induction on the number of digits in the binary representation of natural numbers.

Suppose the result is true for every natural number n such that the number of digits in the binary representation of n is less than or equal to m. Now take any natural number n whose base 2 expansion consists of $m+1$ digits, say,

$$n = a_m 2^m + a_{m-1} 2^{m-1} + \cdots + a_0,$$

where $a_m = 1$ and $a_j \in \{0, 1\}$ for $0 \leq j \leq m-1$. We consider several cases.

Case 1. $a_0 = 0$.

In this case we obtain

$$\begin{aligned} n &= a_m 2^m + a_{m-1} 2^{m-1} + \cdots + a_1 2 \\ &= 2\big(a_m 2^{m-1} + a_{m-1} 2^{m-2} + \cdots + a_1\big). \end{aligned}$$

Hence using the relation for f, we get

$$\begin{aligned} f(n) &= f\big(a_m 2^{m-1} + a_{m-1} 2^{m-2} + \cdots + a_1\big) \\ &= a_1 2^{m-1} + a_2 2^{m-2} + \cdots + a_m \\ &= a_0 2^m + a_1 2^{m-1} + a_2 2^{m-2} + \cdots + a_m. \end{aligned}$$

Case 2. $a_0 = 1, a_1 = 0$.

In this case the binary representation of n is

$$\begin{aligned} n &= a_m 2^m + a_{m-1} 2^{m-1} + \cdots + a_2 2^2 + 1 \\ &= 4\big(a_m 2^{m-2} + a_{m-1} 2^{m-3} + \cdots + a_2\big) + 1 = 4k + 1. \end{aligned}$$

Hence using the expression for $f(4k+1)$, we obtain

$$\begin{aligned} f(n) &= 2f(2k+1) - f(k) \\ &= 2f\big(a_m 2^{m-1} + a_{m-2}2^{m-3} + \cdots + a_2 2 + 1\big) \\ &\qquad - f\big(a_m 2^{m-2} + a_{m-1}2^{m-3} + \cdots + a_2\big) \\ &= 2\big(2^{m-1} + a_2 2^{m-2} + \cdots + a_{m-1}2 + a_m\big) \\ &\qquad - \big(a_2 2^{m-2} + \cdots + a_m\big) \\ &= 2^m + a_2 2^{m-2} + \cdots + a_m \\ &= a_0 2^m + a_1 2^{m-1} + a_2 2^{m-2} + \cdots + a_m, \end{aligned}$$

since $a_0 = 1$ and $a_1 = 0$.

Case 3. $\underline{a_0 = a_1 = 1.}$

In this case, we have

$$n = a_m 2^m + a_{m-1}2^{m-1} + \cdots + a_2 2^2 + 3 = 4k + 3,$$

where $k = a_m 2^{m-2} + a_{m-1}2^{m-3} + \cdots + a_2$. Now using the expression for $f(4k+3)$, we obtain

$$\begin{aligned} f(n) &= 3f(2k+1) - 2f(k) \\ &= 3f\big(a_m 2^{m-1} + a_{m-1}2^{m-2} + \cdots + a_2 2 + 1\big) \\ &\qquad - 2f\big(a_m 2^{m-2} + a_{m-1}2^{m-3} + \cdots + a_2\big) \\ &= 3\big(2^{m-1} + a_2 2^{m-2} + \cdots + a_m\big) \\ &\qquad - 2\big(a_2 2^{m-2} + a_3 2^{m-3} + \cdots + a_m\big) \\ &= 3 \cdot 2^{m-1} + a_2 2^{m-2} + \cdots + a_m \\ &= 2^m + 2^{m-1} + a_2 2^{m-2} + \cdots + a_m \\ &= a_0 2^m + a_1 2^{m-1} + a_2 2^{m-2} + \cdots + a_m. \end{aligned}$$

These 3 cases prove our claim.

Thus the solution to our problem is obtained by counting all those numbers n which do not exceed 1988 and the binary expression for n reads same from left to right as it reads

from right to left. In other words, the binary expression for n should be 'palindromic'. Thus if we write

$$n = a_m 2^m + a_{m-1} 2^{m-1} + \cdots + a_0,$$

then n is 'palindromic' if and only if $a_j = a_{m-j}$ for all indices j, $0 \leq j \leq m$. Note that we must necessarily have $a_m = a_0 = 1$ for such a 'palindromic' $(m+1)$-digit binary expression. Thus we do not have any choice in the selection of the first and last digits. But the remaining digits can be taken either 0 or 1 and the symmetric placing of these digits tell us that there are $2^{[m/2]}$ such numbers. Thus the number of palindromic binary expressions of n digits is $2^{[(n-1)/2]}$.

We observe that $1988 = (11111000100)_2$ in base 2. Thus we need 11 digits to express it. But among all 11 digit symmetric binary expressions, there are two numbers which exceed 1988: one is $(11111111111)_2 = 2047$ and the other is $(11111011111)_2 = 2015$. Thus we have to omit these two while counting palindromic binary expressions which are, when expressed in decimal notation, smaller than 1988. The required number is

$$(1+1+2+2+4+4+8+8+16+16+32) - 2 = 92.$$

■

Several problems use some nice identities which can be built on $\mathbb{N}$. These identities can be diligently exploited to obtain solution(s) of a given problem.

Problem 2.11 Find all $f : \mathbb{N}_0 \to \mathbb{N}_0$ which satisfy

(a) $f(m^2+n^2) = f(m)^2 + f(n)^2$, for all m, n in $\mathbb{N}_0$;

(b) $f(1) > 0$.

Solution: Putting $m = n = 0$ in (a), we get $f(0) = 2f(0)^2$. If $f(0) \neq 0$, then this forces $f(0) = 1/2$, which is impossible since the values taken by f are whole numbers. Thus it follows that $f(0) = 0$ and this in turn leads to $f(m^2) = f(m)^2$. We can write (a) in the form

$$f(m^2 + n^2) = f(m)^2 + f(n)^2 = f(m^2) + f(n^2).$$

We also observe that $f(1) = f(1^2) = f(1)^2$. Since $f(1) > 0$, the only possibility is $f(1) = 1$. This implies that

$$\begin{aligned} f(2) &= f(1^2 + 1^2) = f(1)^2 + f(1)^2 = 1 + 1 = 2; \\ f(4) &= f(2^2) = f(2)^2 = 4; \\ f(5) &= f(2^2 + 1^2) = f(2)^2 + f(1)^2 = 5; \\ f(8) &= f(2^2 + 2^2) = f(2)^2 + f(2)^2 = 8. \end{aligned}$$

Moreover, we see that

$$\begin{aligned} 25 = f(5)^2 &= f(5^2) \\ &= f(3^2 + 4^2) = f(3)^2 + f(4)^2 = f(3)^2 + 16, \end{aligned}$$

so that $f(3) = 3$. (We have to take only nonnegative square root.) This in turn gives

$$\begin{aligned} f(9) &= f(3^2) = f(3)^2 = 9; \\ f(10) &= f(3^2 + 1^2) = f(3)^2 + f(1)^2 = 10. \end{aligned}$$

Using the representation $7^2 + 1^2 = 5^2 + 5^2$, and the known values of $f(5)$ and $f(1)$, we can compute that $f(7) = 7$. Finally, we use the identity $10^2 = 6^2 + 8^2$ to get

$$10^2 = f(10^2) = f(6^2 + 8^2) = f(6)^2 + f(8)^2 = f(6)^2 + 8^2,$$

so that $f(6) = 6$. Thus we have proved that $f(n) = n$ for $n \leq 10$.

We use the following identities:

$$\begin{aligned}
(5k+1)^2+2^2 &= (4k+2)^2+(3k-1)^2;\\
(5k+2)^2+1^2 &= (4k+1)^2+(3k+2)^2;\\
(5k+3)^2+1^2 &= (4k+3)^2+(3k+1)^2;\\
(5k+4)^2+2^2 &= (4k+2)^2+(3k+4)^2;\\
(5k+5)^2 &= (4k+4)^2+(3k+3)^2.
\end{aligned}$$

For $k \geq 3$, we see that each term on the right hand side does not exceed any of the term on the left hand side. This will enable us to proceed by induction in steps of 5. For $k = 2$, we have

$$\begin{aligned}
11^2+2^2 &= 10^2+5^2,\\
12^2+1^2 &= 9^2+8^2,\\
13^2+1^2 &= 11^2+7^2,\\
14^2+2^2 &= 10^2+10^2,\\
15^2 &= 12^2+9^2.
\end{aligned}$$

These identities show that we can compute $f(n)$ by knowing the values of f for smaller numbers. We conclude that $f(n) = n$ for all $n \in \mathbb{N}_0$. ■

There is another useful technique which is often employed successfully to obtain solutions of functional equations. This involves the idea of fixed points. If X is a set and $f : X \to X$ is a mapping then an element $x \in X$ is called a fixed point of f if $f(x) = x$. Existence of fixed points often helps us to solve some problems.

Problem 2.12 Let $\mathbb{N}_0$ denote the set of all nonnegative integers. Find all $f : \mathbb{N}_0 \to \mathbb{N}_0$ satisfying the functional equation

$$f\big(m+f(n)\big) = f\big(f(m)\big)+f(n), \quad \text{for all } m,n \in \mathbb{N}_0. \qquad (1)$$

(IMO-1996)

Solution: Putting $m = n = 0$ in (1), we get $f(0) = 0$. Now taking $m = 0$ in (1) and using $f(0) = 0$, we obtain

$$f\big(f(n)\big) = f(n), \quad \text{for all } n \in \mathbb{N}_0. \tag{2}$$

Plugging this information back in (1), we see that (1) takes the form

$$f\big(m + f(n)\big) = f(m) + f(n), \quad \text{for all } m, n \in \mathbb{N}_0. \tag{3}$$

Conversely, the relation (3) along with the condition $f(0) = 0$ leads to (1). Thus (1) is equivalent to (3) with additional hypothesis that $f(0) = 0$. Thus it is enough if we concentrate on the equation (3).

We observe from (2)(which can be obtained from (3) using $f(0) = 0$) that for each $n \in \mathbb{N}_0$, the element $f(n)$ is a fixed point of f. We assume that f is not an identically zero function. Let α be the least nonzero fixed point of f. Existence of such α is assured by the well ordering principle. If $\alpha = 1$, then the substitution $m = n = 1$ in (3) leads to $f(2) = 2$. By an easy induction we can now prove that $f(n) = n$ for all $n \in \mathbb{N}_0$. Thus identity function is a solution of the given functional equation.

Suppose the least fixed point α of f is larger than 1. Taking $m = n = \alpha$ in (3), we obtain

$$f(2\alpha) = 2\alpha.$$

Again an easy induction gives $f(k\alpha) = k\alpha$, for all natural numbers k. We show that every fixed point of f is precisely of the form $k\alpha$ for some nonnegative integer k. We first observe that sum of any two fixed points of f is also a fixed point of f. Suppose x and y are two fixed points of f so that $f(x) = x$ and $f(y) = y$. Then using (3) we get

$$f(x+y) = f\big(x+f(y)\big) = f\big(f(x)\big)+f(y) = f(x)+y = x+y,$$

showing that $x+y$ is also a fixed point of f. Take an arbitrary fixed point β of f and write β in the form

$$\beta = q\alpha + r,$$

where r is the remainder obtained after dividing β by α; $0 \leq r < \alpha$. Using the fact that f fixes β, we get

$$\beta = f(\beta) = f(q\alpha + r) = f\big(r + f(q\alpha)\big) = f(r) + q\alpha.$$

It follows that $f(r) = r$ showing that r is a fixed point of f. If $r \neq 0$, then r would be a fixed point of f smaller than α. Thus the choice of α forces $r = 0$. We conclude that $\beta = q\alpha$ thus proving our claim that every fixed point of f is an integral multiple of the least nonzero fixed point of f.

We have observed that $f(n)$ is a fixed point of f; thus the set $\{f(n) : n \in \mathbb{N}_0\}$ is a set of fixed points of f. For each $j < \alpha$, we conclude that $f(j) = n_j\alpha$, for some nonnegative integer n_j. We also note that $n_0 = 0$. If n is an arbitrary element of $\mathbb{N}_0$, then we write $n = k\alpha + j$, $0 \leq j < \alpha$. Thus, we see that

$$f(n) = f(j+k\alpha) = f\big(j+f(k\alpha)\big) = f(j)+f(k\alpha) = (n_j+k)\alpha.$$

Conversely, if we define f on $\mathbb{N}_0$ by setting $f(j) = n_j\alpha$, for $0 \leq j < \alpha$, $n_0 = 0$ and $f(n) = (n_j + k)\alpha$ whenever $n = k\alpha + j$, then f is a solution of our functional equation. In fact $f(0) = 0$ is valid. If $n = u\alpha + j$ and $m = v\alpha + k$ are in $\mathbb{N}_0$, then

$$\begin{aligned} f\big(m + f(n)\big) &= f\big(v\alpha + k + f(u\alpha + j)\big) \\ &= f\big(v\alpha + k + (n_j + u)\alpha\big) \\ &= f\big((v + n_j + u)\alpha + k\big) \\ &= (n_k + v + n_j + u)\alpha \\ &= (n_k + v)\alpha + (n_j + u)\alpha \\ &= f(m) + f(n). \end{aligned}$$

We conclude that if $f \not\equiv 0$, then f is of the form

$$f(n) = \left(\left[\frac{n}{\alpha} \right] + n_j \right) \alpha,$$

where

$$n = \left[\frac{n}{\alpha} \right] + j, \quad 0 \leq j < \alpha;$$

here $\alpha \in \mathbb{N}$ is arbitrary and $n_1, n_2, \ldots, n_{\alpha-1}$ are nonnegative integers and $n_0 = 0$. ■

Here are a few interesting problems posed on $\mathbb{N}$ and $\mathbb{Z}$.

Problem 2.13 Find all functions $f : \mathbb{N} \to \mathbb{N}$ which satisfy:

(a) f is a surjective function;

(b) $m|n$ if and only if $f(m)|f(n)$, for any two natural numbers m, n.

Solution: We first show that $f(1) = 1$. Since 1 divides every natural number, (b) shows that $f(1)$ divides $f(n)$ for each $n \in \mathbb{N}$. But (a) shows that $f(n) = 1$ for some natural number n. It follows that $f(1)$ divides 1 and hence $f(1) = 1$.

Suppose $f(m) = f(n)$ for some natural numbers m, n. Then $f(m)|f(n)$ and $f(n)|f(m)$. Now (b) implies that $m|n$ and $n|m$, which in turn leads to $m = n$. Thus f is one-one as well. Since f is given to be an on to function, it is clear that f is a bijection on $\mathbb{N}$. Now (b) can be recast as:

$$\text{(b')} \quad m|n \text{ if and only if } f^{-1}(m)|f^{-1}(n).$$

Next we show that f takes primes to primes. Suppose p is a prime and k divides $f(p)$. Then (b') shows that $f^{-1}(k)$ divides p and hence $f^{-1}(k) = 1$ or p. But then $k = 1$ or $f(p)$ showing that $f(p)$ is a prime. Moreover we show that $f(p^\alpha) = (f(p))^\alpha$ for every prime p and positive integer α. Suppose q is a prime dividing $f(p^\alpha)$. Then $f^{-1}(q)$ is a

prime dividing p^α and hence $f^{-1}(q) = p$. Thus we obtain $q = f(p)$. We infer that the only prime dividing $f(p^\alpha)$ is $f(p)$. We conclude that $f(p^\alpha) = (f(p))^\beta$ for some positive integer β. Since $1, p, p^2, \ldots, p^\alpha$ are distinct divisors of p^α, we see that $1, f(p), f(p^2), \ldots, f(p^\alpha)$ are also distinct divisors of $f(p^\alpha) = (f(p))^\beta$. Thus we obtain the condition, $\alpha \leq \beta$. Considering the function f^{-1}, we can similarly prove that $\beta \leq \alpha$. These two inequalities give $f(p^\alpha) = (f(p))^\alpha$.

Finally we prove that f is multiplicative. That is $f(ab) = f(a)f(b)$ for all natural numbers a, b. If $\gcd(a, b) = 1$, then we observe that $\gcd(f(a), f(b)) = 1$. Indeed, if a prime p divides $\gcd(f(a), f(b))$, then we see that $f^{-1}(p)$ is a prime dividing both a and b contradicting the coprime nature of a and b. Since a and b both divide ab, it follows that $f(a)$ and $f(b)$ divide $f(ab)$. Since $f(a)$ and $f(b)$ are relatively prime, we obtain that $f(a)f(b)$ divides $f(ab)$. Applying the same argument to f^{-1}, we conclude that $f(ab)$ divides $f(a)f(b)$. We thus obtain $f(ab) = f(a)f(b)$ whenever a and b are coprime. Using the prime decomposition of natural numbers, we conclude that

$$f(mn) = f(m)f(n)$$

for all natural numbers m, n.

Thus the structure of a function required in the problem is clear. It must be a bijection on natural numbers taking 1 to 1 and a prime to a prime; it must be a multiplicative function. In particular its restriction to the set P of all primes is a bijection.

Conversely, if we have a bijection g on P, we can construct a function f as follows: $f(1) = 1$, and if n has the prime decomposition $n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_k^{\alpha_k}$ then

$$f(n) = g(p_1)^{\alpha_1} g(p_2)^{\alpha_2} g(p_3)^{\alpha_3} \cdots g(p_k)^{\alpha_k}.$$

We can easily check that f satisfies both (a) and (b). ■

Problem 2.14 Find all functions $h : \mathbb{N} \to \mathbb{N}$ satisfying the functional relation

$$h\big(h(n)\big) + h(n+1) = n + 2, \quad \forall n \in \mathbb{N}. \tag{1}$$

Solution: Put $n = 1$ in (1); we get

$$h\big(h(1)\big) + h(2) = 3.$$

This shows that $h(2) \leq 2$ and $h\big(h(1)\big) \leq 2$. We consider two cases.

Case 1. Suppose $h(2) = 1$ and $h\big(h(1)\big) = 2$. Put $h(1) = k$ so that $h(k) = 2$. Taking $n = 2$ in (1), we obtain

$$h\big(h(2)\big) + h(3) = 4.$$

We thus see that $h(3) = 4 - h(1) = 4 - k$. Since $h(3) \geq 1$, we conclude that $k \leq 3$.

If $k = 1$, then we get

$$2 = h\big(h(1)\big) = h(k) = h(1) = k = 1,$$

which is impossible. If $k = 2$, then again we see that

$$2 = h\big(h(1)\big) = h(k) = h(2) = 1,$$

which is absurd. Finally consider the possibility that $k = 3$. Here we obtain

$$2 = h\big(h(1)\big) = h(k) = h(3) = 4 - k = 4 - 3 = 1,$$

which again is absurd. We conclude that $h(2) = 1$ and $h\big(h(1)\big) = 2$ is not possible.

Case 2. Consider the other possibility that $h(2) = 2$ and $h\big(h(1)\big) = 1$. Taking $n = 2$ in (1), we get

$$h\big(h(2)\big) + h(3) = 4,$$

which leads to $h(3) = 2$. Inductively, we obtain

$$\begin{aligned} h(4) &= 5 - h\big(h(3)\big) = 5 - h(2) = 5 - 2 = 3, \\ h(5) &= 6 - h\big(h(4)\big) = 6 - h(3) = 6 - 2 = 4, \\ h(6) &= 7 - h\big(h(5)\big) = 7 - h(4) = 7 - 3 = 4, \end{aligned}$$

and so on. Thus we see that $h(n) \geq 2$ for $n \geq 2$. Suppose $h(1) = k \geq 2$. Then we obtain

$$3 = h\big(h(1)\big) + h(2) = h(k) + 2 \geq 2 + 2 = 4,$$

which is impossible. Thus $h(1) = 1$ and $h(2) = 2$.

We claim that

$$h(n) = \lfloor n\alpha \rfloor - n + 1,$$

where α is the 'Golden Ratio' defined by $\alpha = \big(1 + \sqrt{5}\big)/2$. We use the following results.

Lemma 1. For each $n \in \mathbb{N}$,

$$\big\lfloor \alpha\big(\lfloor n\alpha \rfloor - n + 1\big) \big\rfloor = n \text{ or } n + 1.$$

Proof: We observe that

$$\begin{aligned} \big\lfloor \alpha\big(\lfloor n\alpha \rfloor - n + 1\big) \big\rfloor &< \alpha(n\alpha - n + 1) \\ &= n(\alpha^2 - \alpha) + \alpha = n + \alpha < n + 2, \end{aligned}$$

and

$$\begin{aligned} \big\lfloor \alpha\big(\lfloor n\alpha \rfloor - n + 1\big) \big\rfloor &> \alpha(n\alpha - 1 - n + 1) - 1 \\ &= n(\alpha^2 - \alpha) - 1 = n - 1. \end{aligned}$$

The result follows. ∎

Lemma 2. For each $n \in \mathbb{N}$,

$$\lfloor (n+1)\alpha \rfloor = \begin{cases} \lfloor n\alpha \rfloor + 2, & \text{if } \lfloor \alpha(\lfloor n\alpha \rfloor - n + 1) \rfloor = n, \\ \lfloor n\alpha \rfloor + 1, & \text{otherwise.} \end{cases}$$

Proof: Obviously $\lfloor (n+1)\alpha \rfloor = \lfloor n\alpha \rfloor + 1$ or $\lfloor n\alpha \rfloor + 2$. Suppose $\lfloor (n+1)\alpha \rfloor = \lfloor n\alpha \rfloor + 1$. Then we get

$$\begin{aligned} \lfloor \alpha(\lfloor n\alpha \rfloor - n + 1) \rfloor &= \lfloor \alpha(\lfloor (n+1)\alpha \rfloor - n) \rfloor \\ &> \alpha(\alpha(n+1) - 1 - n) - 1 = n, \end{aligned}$$

so that $\lfloor \alpha(\lfloor n\alpha \rfloor - n + 1) \rfloor = n+1$ by lemma 1. On the other hand if $\lfloor (n+1)\alpha \rfloor = \lfloor n\alpha \rfloor + 2$, then

$$\begin{aligned} \lfloor \alpha(\lfloor n\alpha \rfloor - n + 1) \rfloor &= \lfloor \alpha(\lfloor (n+1)\alpha \rfloor - n - 1) \rfloor \\ &< \alpha((n+1)\alpha - n - 1) = n + 1. \end{aligned}$$

In view of lemma 1, we get $\lfloor \alpha(\lfloor n\alpha \rfloor - n + 1) \rfloor = n$. ∎

Thus we see that

$$\lfloor (n+1)\alpha \rfloor = \lfloor n\alpha \rfloor + 1 \iff \lfloor \alpha(\lfloor n\alpha \rfloor - n + 1) \rfloor = n + 1;$$

$$\lfloor (n+1)\alpha \rfloor = \lfloor n\alpha \rfloor + 2 \iff \lfloor \alpha(\lfloor n\alpha \rfloor - n + 1) \rfloor = n.$$

To determine h, we use induction on n. We check that

$$h(1) = 1 = \lfloor \alpha \rfloor = \lfloor \alpha \rfloor - 1 + 1,$$

$$h(2) = 2 = 3 - 2 + 1 = \lfloor 2\alpha \rfloor - 2 + 1.$$

Suppose the result is true for $1 \le j \le n$. Using (1), we obtain

$$\begin{aligned} h(n+1) &= n + 2 - h(h(n)) \\ &= n + 2 - h(\lfloor n\alpha \rfloor - n + 1) \\ &= n + 2 - \lfloor \alpha(\lfloor n\alpha \rfloor - n + 1) \rfloor \\ &\quad + (\lfloor n\alpha \rfloor - n + 1) - 1, \end{aligned}$$

since $\lfloor n\alpha \rfloor - n + 1 < 2n - n + 1 = n + 1$. This reduces to

$$h(n+1) = \lfloor n\alpha \rfloor + 2 - \lfloor \alpha(\lfloor n\alpha \rfloor - n + 1) \rfloor.$$

Suppose n is such that $\lfloor \alpha(\lfloor n\alpha \rfloor - n + 1) \rfloor = n$. Then we observe that $\lfloor (n+1)\alpha \rfloor = \lfloor n\alpha \rfloor + 2$ and hence

$$h(n+1) = \lfloor (n+1)\alpha \rfloor - n.$$

If n is such that $\lfloor \alpha(\lfloor n\alpha \rfloor - n + 1) \rfloor = n+1$, then $\lfloor (n+1)\alpha \rfloor = \lfloor n\alpha \rfloor + 1$, and hence we obtain

$$h(n+1) = \lfloor (n+1)\alpha \rfloor + 1 - (n+1) = \lfloor (n+1)\alpha \rfloor - n.$$

Thus in both the cases we see that

$$h(n+1) = \lfloor (n+1)\alpha \rfloor - n.$$

This completes the induction step and we conclude that

$$h(n) = \lfloor n\alpha \rfloor - n + 1,$$

for all n in $\mathbb{N}$. It is easy to verify that the function h thus obtained indeed satisfy the functional equation. ■

Problem 2.15 Let $\mathbb{Z}_3^+$ denote all triples (p, q, r) of non-negative integers. Find all functions $f : \mathbb{Z}_3^+ \to \mathbb{R}$ satisfying

$$\begin{aligned} f(p,q,r) &= 0 \text{ if } pqr = 0; \\ f(p,q,r) &= 1 + \frac{1}{6}\Big[f(p+1,q-1,r) + f(p-1,q+1,r) \\ &\quad + f(p-1,q,r+1) + f(p+1,q,r-1) \\ &\quad + f(p,q+1,r-1) + f(p,q-1,r+1)\Big], \end{aligned}$$

otherwise. (IMO-2001 Short-List)

Solution: We show that there is at most one solution to the problem. Suppose f_1 and f_2 are two such functions. Consider $h = f_1 - f_2$. Then $h : \mathbb{Z}_3^+ \to \mathbb{R}$ satisfies

$$\begin{aligned} h(p,q,r) &= 0 \text{ if } pqr = 0; \\ h(p,q,r) &= \frac{1}{6}\Big[h(p+1,q-1,r) + h(p-1,q+1,r) \\ &\quad + h(p-1,q,r+1) + h(p+1,q,r-1) \\ &\quad + h(p,q+1,r-1) + h(p,q-1,r+1)\Big]. \end{aligned}$$

Consider the part H of the plane $x+y+z = n$ intersecting the octant $\mathbb{R}_3^+ = \{(x,y,z) : x,y,z \geq 0\}$. Suppose h attains its maximum on $H \cap \mathbb{Z}_3^+$ at (p,q,r). We may assume $pqr \neq 0$; for if otherwise $h(p,q,r) = 0$ and hence $h = 0$ on $H \cap \mathbb{Z}_3^+$. We have

$$\begin{aligned} h(p,q,r) &= \frac{1}{6}\Big[h(p+1,q-1,r) + h(p-1,q+1,r) \\ &\quad + h(p-1,q,r+1) + h(p+1,q,r-1) \\ &\quad + h(p,q+1,r-1) + h(p,q-1,r+1)\Big] \\ &\leq h(p,q,r) \end{aligned}$$

It follows that $h(p,q,r) = h(p-1,q+1,r)$. We observe that $(p-1,q+1,r)$ also lies in H. Thus h has its maximum at $(p-1,q+1,r)$. Using this as pivot and repeating the same argument we may reach the point $(p-2,q+2,r)$ which is a point where again h has maximum. Continuing the process we conclude that h has a maximum at $(0,q+p,r)$. However $h(0,q+p,r) = 0$. Thus the maximum of h on $H \cap \mathbb{Z}_3^+$ is 0. Similarly considering $-h$, we can infer that the minimum of h on $H \cap \mathbb{Z}_3^+$ is also 0. Thus $h(a,b,c) = 0$ for all $(a,b,c) \in H \cap \mathbb{Z}_3^+$. Varying n we conclude that $h(a,b,c) = 0$ for all $(a,b,c) \in \mathbb{Z}_3^+$.

Consider the function f defined on $\mathbb{Z}_3^+$ by

$$f(p,q,r) = 0 \text{ if } pqr = 0, \quad f(p,q,r) = \frac{3pqr}{p+q+r} \text{ otherwise.}$$

Then f satisfies both the conditions of the problem.

Thus this function f is the unique solution of the problem. ■

Problem 2.16 Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that

$$f\big(m + f(n)\big) = n + f(m + k), \quad \text{for all } m, n \in \mathbb{N}, \qquad (1)$$

where k is fixed natural number.

Solution: Adding 1 to both sides of (1), we obtain

$$1 + f\big(m + f(n)\big) = n + 1 + f(m + k).$$

Applying f on both sides and using (1), we get

$$m + f(n) + f(1 + k) = m + k + f(n + 1 + k).$$

We can write this in the form

$$f(n + 1 + k) - f(n) = f(1 + k) - k.$$

Taking $n = q(k+1)+r$, where q and r are natural numbers, we obtain

$$f(n) = f\big(q(k + 1) + r\big) = f(r) + q\big(f(k + 1) - k\big). \qquad (2)$$

If $r = 0$, then

$$\begin{aligned} f(n) &= f\big(q(k+1)\big) \\ &= f\big((q-1)(k+1)\big) + \big(f(k+1) - k\big) \\ &= f\big((q-2)(k+1)\big) + 2\big(f(k+1) - k\big) \\ &\vdots \\ &= f(k+1) + (q-1)\big(f(k+1) - k\big). \end{aligned}$$

Let us write

$$f(r) = p(k+1) + t, \quad 0 \le t \le k. \tag{3}$$

We then have

$$\begin{aligned} r + f(m+k) &= f\big(m + f(r)\big) \\ &= f\big(p(k+1) + m + t\big) \\ &= f(m+t) + p\big(f(k+1) - k\big). \end{aligned}$$

If $m = 1$, then

$$r + f(k+1) = f(t+1) + p\big(f(k+1) - k\big). \tag{4}$$

If $2 \le m$, then $m + k = (k+1) + (m-1)$ so that

$$f(m+k) = f(m-1) + \big(f(k+1) - k\big). \tag{5}$$

Let us put $d = k+1-t$. If $2 \le m \le d-1$, then we see that $t+2 \le m+t \le k$ and hence

$$\begin{aligned} r + \big(f(k+1) - k\big) + f(m-1) &= r + f(m+k) \\ &= f(m+t) + p\big(f(k+1) - k\big). \end{aligned} \tag{6}$$

If $m = d$, then $m + t = k + 1$ and we get

$$r + \big(f(k+1) - k\big) + f(m-1) = f(k+1) + p\big(f(k+1) - k\big). \tag{7}$$

If $d+1 \le m \le k+1$, then we see that $k+2 \le m+t \le k+1+t$ and hence

$$f(m+t) = f(m-d) + \big(f(k+1) - k\big).$$

This leads to

$$r + \big(f(k+1) - k\big) + f(m-1) = f(m-d) + \big(p+1\big)\big(f(k+1) - k\big). \tag{8}$$

Summing (4), (6), (7) and (8) over their valid ranges, we obtain

$$(k+1)r + f(k+1) + k\big(f(k+1)-k\big) + \sum_{m=2}^{k+1} f(m-1)$$
$$= \sum_{m=1}^{d-1} f(t+m) + \sum_{m=d+1}^{k+1} f(m-d) + f(k+1)$$
$$+ dp\big(f(k+1)-k\big) + \big(k+1-d\big)\big(p+1\big)\big(f(k+1)-k\big).$$

This simplifies to

$$(k+1)r+k\big(f(k+1)-k\big) = \big[dp+(k+1-d)(p+1)\big]\big(f(k+1)-k\big).$$

A further simplification leads to

$$(k+1)r = \big(f(r)-k\big)\big(f(k+1)-k\big).$$

If we take $r = k+1$, then we obtain $f(k+1)-k = \pm(k+1)$. we can easily rule out the negative sign. Thus $f(k+1)-k = k+1$ giving us $f(r) = k+r$. ■

Problem 2.17 Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ which satisfy the equation

$$f(m+n) + f(m)f(n) = f(mn+1), \qquad (1)$$

for all integers m, n.

Solution: Taking $n = 0$ in (1), we obtain

$$f\big(m\big)\big(1+f(0)\big) = f\big(1\big).$$

If $1 + f(0) \neq 0$, then we see that f is a constant function. But the only constant function satisfying (1) is $f(n) \equiv 0$. If we look for non-constant solutions of (1), then it follows that $1 + f(0) = 0$. This gives

$$f(0) = -1 \quad \text{and} \quad f(1) = 0. \qquad (2)$$

Taking $n = -1$ in (1), we see that

$$f(m-1) + f(m)f(-1) = f(-m+1). \tag{3}$$

If we set $m = 2$ in (3), we obtain $f(-1)(f(2) - 1) = 0$. Thus there are two possibilities: $f(-1) = 0$ or $f(2) = 1$. We consider these separately.

Case 1. Suppose $f(-1) = 0$. Now this forces from (3) that f is an *even* function: $f(-n) = f(n)$ for all integers n. Setting $n = 2$ in (1), we see that

$$f(m+2) + f(m)f(2) = f(2m+1). \tag{4}$$

Similarly, taking $n = -2$ in (1) and using the *even* nature of f we obtain

$$f(m-2) + f(m)f(2) = f(2m-1). \tag{5}$$

Changing m to $m+1$ in (5), we see that

$$f(m-1) + f(m+1)f(2) = f(2m+1). \tag{6}$$

Comparing (4) and (6), we arrive at

$$f(m+2) + f(m)f(2) = f(m-1) + f(m+1)f(2). \tag{7}$$

We treat this as a linear difference equation. Its auxiliary equation is

$$x^3 - f(2)x^2 + f(2)x - 1 = 0.$$

This factors to $(x-1)\left[x^2+(1-f(2))x+1\right] = 0$. Let α and β be solutions of $x^2+(1-f(2))x+1 = 0$. If $f(2) \neq -1, 3$, then $\alpha \neq \beta$ and the general solution of the difference equation (7) is given by

$$f(m) = A + B\alpha^m + C\beta^m, \tag{8}$$

for some constants A, B, C. Using $f(-1) = 0$, $f(0) = -1$ and $f(1) = 0$, we obtain

$$A = \frac{5 - f(2)}{3 - f(2)}, \quad B = C = -\frac{1}{3 - f(2)}. \tag{9}$$

Comparing (4) and (5), we also see that

$$f(m+2) - f(m-2) = f(2m+1) - f(2m-1). \tag{10}$$

Using (8) and (9) in (10), we obtain after some simplification

$$\big(1 + f(2)\big)\big(\alpha^m + \beta^m\big) = \alpha^{2m} + \beta^{2m} \tag{11}$$

If we set $m = 1$ in (11) and use $\alpha+\beta = (f(2)-1)$, $\alpha\beta = 1$, we obtain $f(2) = 0$. Thus (7) leads to the relation $f(m+2) = f(m-1)$ for all $m \in \mathbb{Z}$. Using $f(-1) = 0$, $f(0) = -1$, $f(1) = 0$, $f(2) = 0$, $f(-n) = f(n)$ and induction, we obtain

$$f(3m) = -1, \quad f(3m+1) = 0, \quad f(3m+2) = 0, \quad \text{for all } m \in \mathbb{Z}.$$

Other possibilities are $f(2) = -1$ or $f(2) = 3$.

(i) Suppose $f(2) = -1$. Using this in (4) and (5), we obtain

$$f(m+2) - f(m) = f(2m+1), \tag{12}$$

$$f(m-2) - f(m) = f(2m-1). \tag{13}$$

If we change m to $m+2$ in (13), we get

$$f(m) - f(m+2) = f(2m+3). \tag{14}$$

Now we compare (12) and (14) and obtain $f(2m+1) = -f(2m+3)$ for all integers m. However $f(1) = 0$ and this leads to $f(2m+1) = 0$ for all $m \geq 0$. Invoking the relation $f(-n) = f(n)$, we conclude that $f(2m+1) = 0$ for all integers m. Using this in (12), we obtain $f(m+2) = f(m)$ for

all $m \in \mathbb{Z}$. Since $f(0) = -1$, we conclude that $f(2m) = -1$ for all $m \in \mathbb{Z}$. We thus get the solution

$$f(2m) = -1, \quad f(2m+1) = 0, \quad \text{for all } m \in \mathbb{Z}.$$

(ii) Now consider the possibility $f(2) = 3$. In this case (7) reduces to

$$f(m+2) = 3f(m+1) - 3f(m) + f(m-1). \tag{15}$$

Putting $m = 1$ in (15), we get

$$f(3) = 3f(2) - 3f(1) + f(0) = 9 - 1 = 3^2 - 1.$$

We claim that $f(m) = m^2 - 1$ for all $m \in \mathbb{N}$. This can be easily settled using induction and (15). Since f is *even*, we determine $f(m)$ for all integers:

$$f(m) = m^2 - 1, \quad \text{for all } m \in \mathbb{Z}.$$

Case 2. We now consider the case $f(2) = 1$. Taking $m = n = -1$ in (1), we see that

$$f(-2) + f(-1)^2 = f(2) = 1. \tag{16}$$

Setting $m = 2$, $n = -2$ in (1), we also obtain

$$-1 + f(-2) = f(-3). \tag{17}$$

Using (1), we also obtain $f(2m)+f(m-1)f(m+1) = f\left(m^2\right)$ and $f(2m)+f(m)^2 = f\left(m^2+1\right)$. If we eliminate $f(2m)$ from two relations, we get

$$f\left(m^2+1\right) - f\left(m^2\right) = f(m)^2 - f(m-1)f(m+1). \tag{18}$$

Changing m to $-m$ in (18) and comparing this with (18), we obtain

$$f(-m)^2 - f(-m-1)f(-m+1) = f(m)^2 - f(m-1)f(m+1). \tag{19}$$

Taking $m = 2$ in (19) and using (17), we obtain

$$f(-2)^2 - f(-1)f(-2) + f(-1) - 1 = 0.$$

We solve this for $f(-2)$ to get

$$f(-2) = 1 \quad \text{or} \quad f(-2) = f(-1) - 1.$$

(i) Suppose $f(-2) = 1$. Then (16) shows that $f(-1) = 0$. Using (3) we conclude that f is an *even* function. It follows that $f(2) = f(-2) = 1$. Using this in (7), we see that

$$f(m+2) = f(m+1) - f(m) + f(m-1).$$

Using $f(-1) = 0$, $f(0) = -1$, $f(1) = 0$, $f(2) = 1$ and induction we obtain

$$f(4m) = -1, \ \ f(4m+2) = 1, \ \ f(2m+1) = 0, \ \text{ for all } m \in \mathbb{Z}.$$

(ii) Let us consider the possibility $f(-2) = f(-1) - 1$. In this case (16) gives $f(-1)^2 + f(-1) - 2 = 0$. It follows that $f(-1) = 1$ or $f(-1) = -2$. If $f(-1) = -2$, then $f(-2) = -3$. Taking $n = -2$ in (1) we get

$$f(m-2) - 3f(m) = f(-2m+1). \tag{20}$$

Putting $n = -1$ in (1), we also get

$$f(m-1) - 2f(m) = f(-m+1). \tag{21}$$

Replacing m by $2m$ in (21), we see that

$$f(2m-1) - 2f(2m) = f(-2m+1). \tag{22}$$

It follows that

$$f(2m-1) - 2f(2m) = f(m-2) - 3f(m). \tag{23}$$

But (1) also leads to the relation $f(m+1)+f(2)f(m-1) = f(2m-1)$. If we use this in (23), we get

$$2f(2m) = f(m+1) + 3f(m) + f(m-1) - f(m-2). \quad (24)$$

We use induction and prove that

$$f(m) = m - 1, \quad \text{for all } m \in \mathbb{Z}.$$

If $f(-1) = 1$, then $f(-2) = 0$. Again induction as above leads to

$$f(3m) = -1, \quad f(3m+1) = 0, \quad f(3m+2) = 1, \quad \text{for all } m \in \mathbb{Z}.$$

Thus the solutions are

$$f(n) \equiv 0 :$$

$$f(m) = m^2 - 1;$$

$$f(m) = m - 1;$$

$$f(2m) = -1, \quad f(2m+1) = 0;$$

$$f(3m) = -1, \quad f(3m+1) = 0, \quad f(3m+2) = 0;$$

$$f(4m) = -1, \quad f(4m+2) = 1, \quad f(2m+1) = 0;$$

$$f(3m) = -1, \quad f(3m+1) = 0, \quad f(3m+2) = 1.$$

It is a matter of routine checking that these are indeed solutions of the given functional equations. In each verification, one may need to consider several cases. For example, the last solution may be verified writing each number in one of the form $3n$, $3n+1$, $3n+2$ and considering several cases arising out of various combinations. ■

Exercises

2.1 Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that $f(0) = 1$ and

$$f\big(f(n)\big) = f\big(f(n+2)+2\big) = n, \text{ for all } n \in \mathbb{Z}.$$

2.2 Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that

(a) $f(n)$ is a square for each $n \in \mathbb{N}$;

(b) $f(m+n) = f(m) + f(n) + 2mn$, for all $m, n \in \mathbb{N}$.

2.3 Show that there is no function $f : \mathbb{N}_0 \to \mathbb{N}_0$ such that

$$f\big(f(n)\big) = n + 1987.$$

(IMO 1987)

2.4 Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that

$$f\big(m^2 + f(n)\big) = f(m)^2 + n, \text{ for all } m, n \in \mathbb{N}.$$

2.5 Let $f : \mathbb{N} \to \mathbb{N}$ be a function such that

(a) $f(m) < f(n)$ whenever $m < n$;

(b) $f(2n) = f(n) + n$ for all $n \in \mathbb{N}$; and

(c) n is a prime number whenever $f(n)$ is a prime number.

Find $f(2001)$.

2.6 Let $f, g : \mathbb{N}_0 \to \mathbb{N}_0$ be such that

(a) $f(1) > 0$, $g(1) > 0$;

(b) $f\big(g(n)\big) = g\big(f(n)\big)$ for all n in $\mathbb{N}_0$;

(c) $f\big(m^2 + g(n)\big) = f(m)^2 + g(n)$ for all m, n in $\mathbb{N}_0$;

(d) $g\big(m^2 + f(n)\big) = g(m)^2 + f(n)$ for all m, n in $\mathbb{N}_0$.

Prove that $f(n) = g(n) = n$ for all n in $\mathbb{N}_0$.

2.7 Determine whether or not there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that

(a) $f(1) = 2$;

(b) $f(f(n)) = f(n) + n$, for all natural numbers n;

(c) $f(n) < f(n+1)$ for all $n \in \mathbb{N}$.

(IMO-1993)

2.8 Find all functions $f : \mathbb{N} \to \mathbb{N}$ satisfying the relation

$$f\Big(f\big(f(n)\big)\Big) + f\big(f(n)\big) + f(n) = 3n,$$

for all natural numbers n.

2.9 Find all $f : \mathbb{Z} \to \mathbb{Z}$ which satisfy the conditions $f(1) = 1$ and

$$f\big(m+n\big)\big\{f(m) - f(n)\big\} = f\big(m-n\big)\big\{f(m) + f(n)\big\}$$

for all integers m, n.

2.10 Let $f : \mathbb{N} \to \mathbb{N}$ be a function defined by $f(1) = 1$ and

$$f(n+1) = \begin{cases} f(n) + 2 & \text{if } f\big(f(n) - n + 1\big) = 2, \\ f(n) + 1 & \text{otherwise .} \end{cases}$$

Prove that $f(n) = [n\alpha]$, where $\alpha = \dfrac{1+\sqrt{5}}{2}$ is the 'golden ratio' and $[x]$ denotes the integral part of x.

2.11 Find all bounded functions $f : \mathbb{Z} \to \mathbb{Z}$ such that

$$f(m+n) + f(m-n) = 2f(m)f(n), \text{ for all integers } m, n.$$

2.12 Find all $f : \mathbb{N}_0 \to \mathbb{N}_0$ which obey the functional equation

$$2f(m^2 + n^2) = f(m)^2 + f(n)^2$$

for all nonnegative integers m, n.

(Korean Mathematical Olympiad-1998)

2.13 Find a bijection $f : \mathbb{N}_0 \to \mathbb{N}_0$ such that

$$f(3mn + m + n) = 4f(m)f(n) + f(m) + f(n),$$

for all m, n in $\mathbb{N}_0$. (IMO-1996 Short-List)

2.14 Find all functions $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ which satisfy

(a) $f(n, n) = n$ for all $n \in \mathbb{N}$;

(b) $f(n, m) = f(m, n)$ for all $n, m \in \mathbb{N}$;

(c)

$$\frac{f(m, n + m)}{f(m, n)} = \frac{m + n}{n}, \quad \forall m, n \in \mathbb{N}.$$

(AMM-1988)

2.15 Find all $f : \mathbb{N} \to \mathbb{N}$ such that

$$f(n) + f(n + 1) = f(n + 2)f(n + 3) - k,$$

where $k = p - 1$ for some prime p.

2.16 Let $f : \mathbb{N} \to \mathbb{N}$ be a function defined by

$$f(1) = 2, \quad f(2) = 1, \quad f(3n) = 3f(n),$$

$$f(3n + 1) = 3f(n) + 2, \quad f(3n + 2) = 3f(n) + 1.$$

Find the number of $n \leq 2001$ for which $f(n) = 2n$.

2.17 Let p be a given odd prime. Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ satisfying the following conditions:

(a) If $m \equiv n \pmod{p}$ for $m, n \in \mathbb{Z}$, then $f(m) = f(n)$;

(b) $f(mn) = f(m)f(n)$ for all $m, n \in \mathbb{Z}$.

2.18 Consider all functions $f : \mathbb{N} \to \mathbb{N}$ satisfying

$$f\big(m^2 f(n)\big) = n\big(f(m)\big)^2,$$

for all $m, n \in \mathbb{N}$. Determine the least possible value of $f(1998)$.

(IMO-1998)

2.19 Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ which satisfy the equation

$$f\big(a^3 + b^3 + c^3\big) = f(a)^3 + f(b)^3 + f(c)^3,$$

for all integers a, b, c. (AMM-1999)

2.20 The set of all positive integers is the union of two disjoint subsets:

$$\mathbb{N} = \{f(1), f(2), \ldots, f(n), \ldots\} \cup \{g(1), g(2), \ldots, g(n), \ldots\},$$

where

$$f(1) < f(2) < \cdots < f(n) \cdots,$$

$$g(1) < g(2) < \cdots < g(n) \cdots,$$

and

$$g(n) = f\big(f(n)\big) + 1, \quad \text{for all } n \geq 1.$$

Determine $f(240)$. (IMO-1978)

2.21 The function $f(n)$ is defined for all positive integers n and takes on nonnegative integer values. Also, for all m, n

$$f(m+n) - f(m) - f(n) = 0 \text{ or } 1,$$

$$f(2) = 0, \ f(3) > 0, \ \text{ and } f(9999) = 3333.$$

Find $f(1982)$. (IMO-1982)

2.22 Find all $f : \mathbb{N}_0 \to \mathbb{N}_0$ which satisfy the equation

$$f\big(f(m)^2 + f(n)^2\big) = m^2 + n^2, \ \text{ for all } m, n \in \mathbb{N}_0.$$

2.23 Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that

$$f(m+n) + f(mn-1) = f(m)f(n) + 2,$$

for all integers m, n.

2.24 For which integers k, there exists a function $f : \mathbb{N} \to \mathbb{Z}$ which satisfies

(a) $f(1995) = 1996$, and

(b) $f(xy) = f(x) + f(y) + kf\big(\gcd(x,y)\big)$, for all $x, y \in \mathbb{N}$?

(Czech-Slovak Contest, 1996-97)

2.25 Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that

$$f(m+n) + f(mn-1) = f(m)f(n),$$

for all $m, n \in \mathbb{Z}$.

2.26 Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ which obey the equation

$$f(m+n) + f(mn) = f(m)f(n) + 1,$$

for all $m, n \in \mathbb{Z}$.

3

Equations on Real line

We have seen in the earlier chapter how the inherent properties of the set of all natural numbers have been advantageously used to capture the solutions of functional equations on $\mathbb{N}$. In particular two popular themes we have made use are: (a) the Principle of Mathematical Induction and (b) the absence of natural number between n and $n+1$. Thus there is an immediate successor for every natural number. We have also seen how another variation of induction, namely, well ordering principle can be used in some problems. In the case of $\mathbb{Z}$, although this well ordering fails, the property that each integer has an immediate predecessor or an immediate successor still remains valid and form the basis of solution for many functional equations on $\mathbb{Z}$.

However, these properties totally fail when we consider 'Real Number System'. We have had a taste of this in the problem 5 of chapter 2. Since the range of the function was given to be $[1, \infty)$, we could not fix the value of $f(n)$ unlike in problems 1 and 2 there. We had to use different strategy to fix the value of f. This is the general difficulty while dealing with functional equations on $\mathbb{R}$. We stress again that there is no general uniform method which assures a solution of the given problem. Each problem should be dealt purely on its merits. Again, the properties of domain and range play an extremely crucial role in getting the solutions. In particular, we make use of addition, multiplication, existence of inverse for any nonzero real number, ordering on $\mathbb{R}$ and a crucial fact that the square of a real number is nonnegative.

Problem 3.1 Determine all functions $f : \mathbb{R} \setminus \{0, 1\} \to \mathbb{R}$

which satisfy the equation

$$f(x) + f\left(\frac{1}{1-x}\right) = \frac{2(1-2x)}{x(1-x)}, \tag{1}$$

valid for all $x \neq 0$ and $x \neq 1$.

Solution: Let us put $y = \dfrac{1}{1-x}$, so that (1) takes the form

$$f(x) + f(y) = 2\left(\frac{1}{x} - y\right). \tag{2}$$

Replacing x by y in (1), we get

$$f(y) + f\left(\frac{1}{1-y}\right) = \frac{2(1-2y)}{y(1-y)}.$$

Putting $z = \dfrac{1}{1-y}$, this reduces to

$$f(y) + f(z) = 2\left(\frac{1}{y} - z\right). \tag{3}$$

Now if we replace x by z in (1) and use the fact that

$$\frac{1}{1-z} = \frac{y-1}{y} = x,$$

we obtain

$$f(z) + f(x) = 2\left(\frac{1}{z} - x\right). \tag{4}$$

Adding (2) and (4), we get

$$2f(x) + f(y) + f(z) = 2\left(\frac{1}{x} - y\right) + 2\left(\frac{1}{z} - x\right).$$

If we use (3) here, the relation changes to

$$2f(x) + 2\left(\frac{1}{y} - z\right) = 2\left(\frac{1}{x} - y\right) + 2\left(\frac{1}{z} - x\right).$$

This simplifies to

$$f(x) = \left(\frac{1}{x} - x\right) + \left(\frac{1}{z} + z\right) - \left(\frac{1}{y} + y\right).$$

But we observe that

$$y + \frac{1}{y} = \frac{1}{1-x} + 1 - x, \quad z + \frac{1}{z} = \frac{x-1}{x} + \frac{x}{x-1}.$$

Using these relations, we conclude that

$$f(x) = \frac{x+1}{x-1}.$$

We easily verify that this is indeed the solution. ■

Problem 3.2 Find all functions $f : \mathbb{R} \to \mathbb{R}$ which obey the equation

$$f\big((x-y)^2\big) = f(x)^2 - 2xf(y) + y^2. \tag{1}$$

Solution: One can easily guess that $f(x) = x$ is a solution of this functional equation. Are there any other solutions which are not obvious but hidden in the equation? Indeed there is one more solution, $f(x) = x+1$, which is not apparent from the equation. We see that for the function $f(x) = x + 1$, we have

$$f\big((x-y)^2\big) = (x-y)^2 + 1 = x^2 - 2xy + y^2 + 1,$$

and

$$\begin{aligned} f(x)^2 - 2xf(y) + y^2 &= (x+1)^2 - 2x(y+1) + y^2 \\ &= x^2 - 2xy + y^2 + 1. \end{aligned}$$

How do we capture these two and others if any?

Put $y = 0$ in (1) to obtain

$$f(x^2) = f(x)^2 - 2xf(0), \tag{2}$$

and put $x = 0$ to get

$$f(y^2) = f(0)^2 + y^2. \tag{3}$$

Taking $y = 0$ in (3) we see that $f(0)^2 = f(0)$ giving $f(0) = 0$ or $f(0) = 1$. Taking $x = y$ in (1), we obtain

$$f(0) = f(x)^2 - 2xf(x) + x^2 = \big(f(x) - x\big)^2.$$

If $f(0) = 0$, then the above relation shows that $f(x) = x$ for all $x \in \mathbb{R}$. If $f(0) = 1$, then $f(x) - x = \pm 1$ and hence $f(x) = x \pm 1$. Which sign should we choose here? It may also happen that $f(x) = x + 1$ for some real number x and $f(y) = y - 1$ for some other real number y. We have to resolve this before concluding any thing.

Suppose $f(x_0) = x_0 - 1$ for some real number x_0. Then using (3) and (2),we get

$$1+x_0^2 = f\big(x_0^2\big) = f(x_0)^2 - 2x_0 = (x_0-1)^2 - 2x_0 = x_0^2 - 4x_0 + 1.$$

This forces $x_0 = 0$. But then we obtain

$$1 = f(0) = f(x_0) = x_0 - 1 = -1,$$

which is absurd. We conclude that $f(x) = x + 1$ for all real numbers x

It follows that $f(x) = x$ and $f(x) = x + 1$ are the only solutions of the given functional equation. ∎

Problem 3.3 Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

(a) $f(-x) = -f(x)$ for all real x;

(b) $f(x + 1) = f(x) + 1$, for all real x;

(c) $f\left(\dfrac{1}{x}\right) = \dfrac{f(x)}{x^2}$ for all $x \neq 0$.

Solution: Putting $x = 0$ in (a), we obtain $f(0) = 0$. Using (b), we see that $f(1) = 1$. An easy induction using (b) shows that $f(n) = n$ for all natural numbers n. Another application of (a) now implies that $f(n) = n$ for all integers n.

Consider $1 + \dfrac{1}{x}$, for all $x \neq 0$ and $x \neq -1$. Using (b) and (c), we obtain

$$f\left(1 + \frac{1}{x}\right) = 1 + f\left(\frac{1}{x}\right) = 1 + \frac{f(x)}{x^2}.$$

On the other hand, we write

$$1 + \frac{1}{x} = \frac{x+1}{x} = \frac{1}{(x/(x+1))},$$

and this gives in view of (c) another expression

$$f\left(1 + \frac{1}{x}\right) = f\left(\frac{1}{x/(x+1)}\right) = \frac{f(x/(x+1))}{(x/(x+1))^2}.$$

But we also have

$$\begin{aligned} f\left(\frac{x}{x+1}\right) &= f\left(1 - \frac{1}{x+1}\right) \\ &= 1 - f\left(\frac{1}{x+1}\right) \\ &= 1 - \frac{f(x+1)}{(x+1)^2} \\ &= \frac{(x+1)^2 - 1 - f(x)}{(x+1)^2}. \end{aligned}$$

Using this expression, we obtain

$$f\left(1 + \frac{1}{x}\right) = \frac{(x+1)^2 - 1 - f(x)}{x^2}.$$

Now comparing two expressions for $f\left(1+\frac{1}{x}\right)$, we see that

$$x^2 + f(x) = x^2 + 2x - f(x), \quad \text{for all } x \neq 0, x \neq -1.$$

Solving for $f(x)$, we conclude that

$$f(x) = x, \quad \text{for all } x \neq 0, x \neq -1.$$

But we know that $f(0) = 0$ and $f(-1) = -f(1) = -1$. Thus $f(x) = x$ holds good for all real numbers x. ■

Problem 3.4 Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that

(a) $f(x+y) = f(x) + f(y)$ for all x, y in $\mathbb{R}$;

(b) $f\left(\frac{1}{x}\right) = \frac{f(x)}{x^2}$ for all $x \neq 0$.

Prove that $f(x) = cx$ for all $x \in \mathbb{R}$, for some constant c.

Solution: It is easy to check that (a) gives $f(0) = 0$ and $f(-x) = -f(x)$ for all real x. We know, for $x \neq 0$ and $x \neq 1$, the identity

$$\frac{1}{x-1} - \frac{1}{x} = \frac{1}{x(x-1)}.$$

This in conjunction with the property (a) gives

$$f\left(\frac{1}{x-1}\right) - f\left(\frac{1}{x}\right) = f\left(\frac{1}{x(x-1)}\right).$$

Now an application of (b) yields

$$\frac{f(x-1)}{(x-1)^2} - \frac{f(x)}{x^2} = \frac{f\big(x(x-1)\big)}{x^2(x-1)^2}.$$

This simplifies to

$$x^2 f(x-1) - (x-1)^2 f(x) = f\big(x^2 - x\big).$$

If we use (a) and $f(-y) = -f(y)$ here, we obtain

$$f(x^2) + x^2 f(1) = 2xf(x).$$

Replacing x by $x + (1/x)$ and simplifying, we obtain

$$f(x) = \left(\frac{f(2) + 2f(1)}{4}\right) x,$$

valid for all $x \neq 0$ and $x \neq 1$. Putting $x = 2$ in this relation, we see that $f(2) = 2f(1)$. Thus we obtain $f(x) = f(1)x$, for all $x \neq 0$ and $x \neq 1$. This remains valid for $x = 0$ and $x = 1$ as may be seen by inspection. ■

The above problems reveal the fact that using simple manipulations, we can solve some functional equations on $\mathbb{R}$. We have not effectively used any structure of $\mathbb{R}$ to arrive at the solution. Next few problems tell us how to use the known structure(s) of real numbers to solve equations.

Problem 3.5 Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that

(a) $f(x + y) = f(x) + f(y)$, for all real numbers x, y;

(b) $f(xy) = f(x)f(y)$, for all real numbers x, y.

Show that $f(x) = 0$ for all reals x or $f(x) = x$ for all reals x.

Solution: The result says that the only functions on $\mathbb{R}$ which preserve both addition and multiplication is either simply the 'zero' function or the 'identity' function.

We first show that $f(rx) = rf(x)$ for all rationals r and reals x. Taking $x = y = 0$ in (a), we get $f(0) = 0$. Taking $y = -x$ in (a) and using $f(0) = 0$, we obtain $f(-x) = -f(x)$ for all $x \in \mathbb{R}$. Putting $y = x$ in (a), we get $f(2x) = 2f(x)$, for all real numbers x. An easy induction using (a) yields $f(nx) = nf(x)$ for all $n \in \mathbb{N}$ and $x \in \mathbb{R}$. Since $f(-x) = -f(x)$, we conclude that $f(nx) = nf(x)$ for all integers n

and all reals x. Take any rational $r = p/q$, where p is an integer and q is a natural number. Using the informations obtained so far, we see that

$$pf(x) = f(px) = f(qrx) = qf(rx).$$

It follows that $f(rx) = rf(x)$ for all $x \in \mathbb{R}$ and $r \in \mathbb{Q}$. In particular, we get $f(r) = rf(1) = cr$ for all rational numbers r, where $c = f(1)$ is a constant.

We have not yet made use of (b). Taking $x = y = 1$ in (b), we get $f(1) = f(1)^2$, forcing $f(1) = 0$ or $f(1) = 1$. If $f(1) = 0$, then taking $y = 1$ in (b) we see that $f(x) = 0$ for all real numbers x. Thus we may assume that $f(1) = 1$. Putting $y = x$ in (b), we observe that $f(x^2) = f(x)^2$ for all real numbers x. But we know that $z^2 \geq 0$ for all real z. Thus if $z \geq 0$, then $\sqrt{z}$ is a real number and

$$f(z) = f((\sqrt{z})^2) = f\left(\sqrt{z}\right)^2 \geq 0.$$

We conclude that f maps a nonnegative real number to another nonnegative real number. This property stems out as a consequence of the structure of real line and leads to an unexpected bonanza.

Take any two real numbers a, b with $a < b$. Then $b - a > 0$ and hence $f(b - a) \geq 0$. But then (a) and the fact that $f(-x) = -f(x)$ give

$$0 \leq f(b - a) = f(b) - f(a),$$

so that $f(a) \leq f(b)$. In other words we have proved that f is a nondecreasing function on $\mathbb{R}$ using (a) and (b). This and the fact $f(r) = rf(1) = r$ for all rationals lead to the solution of our problem.

We claim $f(x) = x$ for all $x \in \mathbb{R}$. Suppose $f(x) < x$ for some x. Now choose a rational r such that $f(x) < r < x$. Here we are making use of another important structure of real line; there is a rational between any two distinct

real numbers. Since f is nondecreasing, we conclude that $f(r) \leq f(x)$. But $f(r) = r$ and hence we get $r \leq f(x)$. This contradicts $f(x) < r$.

Similarly, we can prove that $x < f(x)$ is also untenable. We conclude that $f(x) = x$ for all real x. ■

The above problem shows that the inherent structure of $\mathbb{R}$ helps a lot to decide the solution of some functional equations. We have used the multiplicative nature of the function to conclude that such a function maps nonnegative reals to nonnegative reals. Basic to such conclusion is the fact that the square of a real number is nonnegative. The additivity condition then implies that the function is nondecreasing. Since any additive function on $\mathbb{R}$ is uniquely determined on rational numbers, we can now determine the value of the function at every real number. We relied here on another important property of $\mathbb{R}$; between any two real numbers, we can always find a rational number. This is known as the *density* of $\mathbb{Q}$ in $\mathbb{R}$. Yet another important property of real numbers is the *Law of Trichotomy*; given any two real numbers x and y, there prevails one and only one relation between them, namely, $x < y$ or $x = y$ or $x > y$. The following problem also shows how all these ideas can be effectively used in the solution of a functional equation.

Problem 3.6 Find all functions $f : \mathbb{R} \to \mathbb{R}$ which satisfy the equation

$$f\big(x^2 + f(y)\big) = f(x)^2 + y, \tag{1}$$

for all $x, y \in \mathbb{R}$.

(IMO-1992)

Solution: Taking $x = 0$ in (1) and putting $f(0) = s$, we get

$$f\big(f(y)\big) = s^2 + y, \quad \text{for all } y \in \mathbb{R}. \tag{2}$$

Similarly taking $y = 0$ in (1), we obtain

$$f(x^2 + s) = f(x)^2, \quad \text{for all } x \in \mathbb{R}. \tag{3}$$

Setting $x = 0$ in (3) leads to the relation,

$$f(s) = s^2. \tag{4}$$

Addition of (3) and (4) gives

$$s^2 + f(x^2 + s) = f(x)^2 + f(s).$$

This implies that

$$f(s^2 + f(x^2 + s)) = f(f(x)^2 + f(s)).$$

Using (1), we reduce the above relation to

$$f(s)^2 + x^2 + s = \Big(f(f(x))\Big)^2 + s.$$

If we now use (2) and (4), we see that

$$s^4 + x^2 + s = (s^2 + x)^2 + s.$$

This simplifies to $2s^2x = 0$, valid for all $x \in \mathbb{R}$, which is possible only if $s = 0$. Using this fact in (2) and (3) , we get

$$f(f(y)) = y, \quad \text{for all } y \in \mathbb{R}, \tag{5}$$

and

$$f(x^2) = f(x)^2, \quad \text{for all } x \in \mathbb{R}. \tag{6}$$

We observe that (6) implies $f(x) \geq 0$ if $x \geq 0$. If $f(x) = 0$ for some x, then

$$f(x^2) = f(x^2 + f(x)) = f(x)^2 + x = x,$$

so that $x = f(x^2) = f(x)^2 = 0$. It follows that $f(x) > 0$ if $x > 0$.

Replacing x by $f(x)$ in (1), we get

$$f\big(f(x)^2 + f(y)\big) = \Big(f\big(f(x)\big)\Big)^2 + y = x^2 + y.$$

This in turn gives

$$f\big(x^2 + y\big) = f(x)^2 + f(y) = f\big(x^2\big) + f(y).$$

Thus we get a restricted form of additivity; $f(z+y) = f(z)+f(y)$ for all $z \geq 0$ and all real y.

Suppose we take two real numbers x, y such that $x > y$. Then $x - y > 0$ and hence

$$f(x) = f(x - y + y) = f(x - y) + f(y) > f(y);$$

we have used the fact that $x - y > 0$ and the restricted additivity which we have proved in the earlier paragraph. We thus obtain a property of f that it is strictly increasing on $\mathbb{R}$. This is enough to fix the values of f. If $f(x) > x$ for some x, then the strictly increasing nature of f gives $f(f(x)) > f(x)$. But f is involutive; i.e., $f(f(x)) = x$ for all $x \in \mathbb{R}$. We thus arrive at $x > f(x)$ contradicting what we have started with. Similarly, we can easily check that $f(x) < x$ is also not possible. The only left-out option is $f(x) = x$ for all $x \in \mathbb{R}$. It is easy to verify that this function satisfies the given equation.

Alternate Solution: We see from the given equation that $f\big(f(y)\big) = y + \big(f(0)\big)^2$. Suppose $f(y) < y$ for some y. Then we can find x such that $y - f(y) = x^2$. This leads to

$$f(y) = f\big(x^2 + f(y)\big) = y + f(x)^2,$$

showing that $y \leq f(y)$. It follows that

$$y \leq f(y), \quad \text{for all } y \in \mathbb{R}.$$

Now choose $y_0 < -f(0)^2$ and consider $\alpha = f(y_0)$. We see that

$$\alpha \leq f(\alpha) = f\big(f(y_0)\big) = y_0 + f(0)^2 < 0.$$

Thus α, $f(\alpha)$ are both negative and $\alpha \leq f(\alpha)$. It follows that $f(\alpha)^2 \leq \alpha^2$. Take any $x \in \mathbb{R}$. We observe that

$$\alpha^2 + x \leq \alpha^2 + f(x) \leq f\big(\alpha^2 + f(x)\big) = x + f(\alpha)^2 \leq x + \alpha^2.$$

Thus there is equality throughout and this gives $f(x) = x$.

∎

Problem 3.7 Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function such that

$$f\left(\frac{x+y}{x-y}\right) = \frac{f(x)+f(y)}{f(x)-f(y)}, \tag{1}$$

for all $x \neq y$. Prove that $f(x) = x$ for all $x \in \mathbb{R}$.

Solution: We use similar techniques here as in the earlier problem but in a more subtle way. We start with the observation that such a function is one-one and hence cannot be constant on any interval. Otherwise the right hand side is not defined since the denominator reduces to zero. Taking $y = 0$ in (1), we obtain

$$f(1) = \frac{f(x)+f(0)}{f(x)-f(0)}.$$

This can be solved for $f(x)$ to get

$$f(x)\big(f(1)-1\big) = f(0)\big(f(1)+1\big).$$

If $f(1) \neq 1$, then we get

$$f(x) = \frac{f(0)\big(f(1)+1\big)}{f(1)-1},$$

showing that f is a constant function. Since we have ruled out constant functions, we conclude that $f(1) = 1$ and hence $f(0) = 0$. Now replacing y by $x - 2$ in (1), we obtain

$$f(x-1) = \frac{f(x)+f(x-2)}{f(x)-f(x-2)}. \tag{2}$$

If we replace x by $x-1$ and y by 1 in (1), we get

$$f\left(\frac{x}{x-2}\right) = \frac{f(x-1)+1}{f(x-1)-1}, \tag{3}$$

where we have used $f(1) = 1$. If we use the value of $f(x-1)$ from (2) in (3) and simplify, we get

$$f\left(\frac{x}{x-2}\right) = \frac{f(x)}{f(x-2)}. \tag{4}$$

A comparison of (3) and (4) shows that

$$f(x) = f(x-2)\left\{\frac{f(x-1)+1}{f(x-1)-1}\right\}. \tag{5}$$

Putting $x = 3$ in (3), we get

$$f(3) = \frac{f(2)+1}{f(2)-1}.$$

Similarly, the substitution $x = 4$ in (4) leads to $f(4) = f(2)^2$. Taking $x = 5$ in (5), we also obtain

$$\begin{aligned} f(5) &= f(3)\left\{\frac{f(4)+1}{f(4)-1}\right\} \\ &= \left\{\frac{f(2)+1}{f(2)-1}\right\}\left\{\frac{f(2)^2+1}{f(2)^2-1}\right\} \\ &= \frac{f(2)^2+1}{(f(2)-1)^2}. \end{aligned}$$

However we can also express $f(5)$ in a different way using (1):

$$f(5) = f\left(\frac{3+2}{3-2}\right) = \frac{f(3)+f(2)}{f(3)-f(2)}.$$

Using the expression for $f(3)$, which we have obtained earlier, we get

$$f(5) = \frac{f(2)^2+1}{1+2f(2)-f(2)^2}.$$

Comparing two expressions for $f(5)$, we see that,

$$(f(2)-1)^2 = 1 + 2f(2) - f(2)^2.$$

This quadratic equation for $f(2)$ simplifies to $f(2)^2 = 2f(2)$. We conclude that $f(2) = 0$ or $f(2) = 2$. Since f is one-one and $f(0) = 0$, we cannot have $f(2) = 0$. The only possibility is $f(2) = 2$.

This is the most difficult and important step in getting a solution of our problem. The rest follows familiar track. We compute $f(3) = 3, f(4) = 4$ and $f(5) = 5$. Suppose $f(k) = k$ for all natural numbers $k \leq n$, where n is a natural number. Then (5) shows that

$$f(n+1) = f(n-1)\left\{\frac{f(n)+1}{f(n)-1}\right\}.$$

Since $f(n-1) = n-1$ and $f(n) = n$, we obtain $f(n+1) = n+1$. We conclude that $f(n) = n$ for all natural numbers n.

Replacing y by xz in (1), we get

$$f\left(\frac{x+xz}{x-xz}\right) = \frac{f(x)+f(xz)}{f(x)-f(xz)}.$$

But we also see that

$$f\left(\frac{x+xz}{x-xz}\right) = f\left(\frac{1+z}{1-z}\right) = \frac{1+f(z)}{1-f(z)},$$

where we have used (1) again. Comparing these two expressions and solving for $f(xz)$, we obtain

$$f(zx) = f(z)f(x). \tag{6}$$

A priori this is valid for $x \neq 0$ and $z \neq 1$. But since $f(0) = 0$ and $f(1) = 1$, we see that this multiplicative property is valid for all x, z in $\mathbb{R}$. Taking $y = -x$ in (1), we see that

$$f(0) = \frac{f(x)+f(-x)}{f(x)-f(-x)},$$

giving us $f(-x) = -f(x)$. Thus f is also an odd function. Since $f(n) = n$ for all natural numbers n, now it follows that $f(k) = k$ for all integers k. This with multiplicativity (6) implies that $f(r) = r$ for all rational numbers r. Since (6) implies that $f(x^2) = f(x)^2$, it follows that f maps non-negative reals to nonnegative reals. Since f is one-one and $f(0) = 0$, we conclude that $f(x) > 0$ whenever $x > 0$.

Suppose $x > y$. We consider different cases:

(a) Suppose $x > y \geq 0$. Here we obtain

$$\frac{f(x) + f(y)}{f(x) - f(y)} = f\left(\frac{x+y}{x-y}\right) > 0,$$

showing that $f(x) > f(y)$.

(b) Suppose $y < 0 < x$. In this case $f(y) < 0$ and $f(x) > 0$ so that $f(y) < f(x)$.

(c) Consider the case $y < x < 0$. Then $0 < -x < -y$ and by (a), we conclude that $f(-x) < f(-y)$. Using the fact that f is an odd function, this reduces to $f(y) < f(x)$

It follows that f is a strictly increasing function on $\mathbb{R}$. Since $f(r) = r$ for all rational numbers r, we obtain $f(x) = x$ for all real numbers x. ■

Problem 3.8 Find all functions $f : \mathbb{R} \to \mathbb{R}$ which satisfy the functional equation

$$f\big(xf(x) + f(y)\big) = f(x)^2 + y, \quad \text{for all } x, y \in \mathbb{R}. \tag{1}$$

Solution: Let us put $f(0) = s$. Taking $x = 0$ in (1), we see that

$$f\big(f(y)\big) = s^2 + y. \tag{2}$$

This shows that f is an on to function: replace y by $y - s^2$. Thus we can find some real a such that $f(a) = 0$. Taking $x = a$ in (1), we obtain

$$f\big(f(y)\big) = y, \quad \text{for all } y \in \mathbb{R}. \tag{3}$$

If $f(x) = f(y)$ for some reals x, y, then (3) shows that $x = y$. Thus f is also an one-one function. Comparing (2) and (3), we also conclude that $s = 0$; i.e., $f(0) = 0$. Taking $y = 0$ in (1), we obtain

$$f\big(xf(x)\big) = f(x)^2, \quad \text{for all } x \in \mathbb{R}. \tag{4}$$

Replacing x by $f(x)$ in (4) and using (3), we obtain

$$f\big(xf(x)\big) = x^2, \quad \text{for all } x \in \mathbb{R}. \tag{5}$$

Comparing (4) and (5), we conclude that $f(x)^2 = x^2$ for all real numbers x. Thus we obtain $f(x) = \pm x$.

Can it happen that $f(x) = x$ for some x and $f(y) = -y$ for some $y \neq x$? Suppose there are $x \neq 0$ and $y \neq 0$ such that $f(x) = x$ and $f(y) = -y$. Using (1), we obtain

$$f\big(x^2 - y\big) = x^2 + y.$$

Thus we see that $\pm\big(x^2 - y\big) = x^2 + y$. But this forces either $x = 0$ or $y = 0$. We conclude that $f(x) = x$ for all x or $f(x) = -x$ for all x. It is easy to verify that $f(x) = x$ and $f(x) = -x$ are indeed solutions. ∎

Problem 3.9 Find all $f : \mathbb{R} \to \mathbb{R}$ such that

$$f\big(f(x) + y\big) = f\big(x^2 - y\big) + 4f(x)y, \tag{1}$$

for all $x, y \in \mathbb{R}$.

Solution: It is easy to check that $f(x) \equiv 0$ and $f(x) = x^2$ are solutions of this problem. We show that these are the only solutions of the problem.

Suppose $f(a) \neq a^2$ for some a. Replacing y in (1) by $\big(x^2 - f(x)\big)/2$, we get

$$f(x)\big(x^2 - f(x)\big) = 0.$$

Since $f(a) \neq a^2$, it follows that $f(a) = 0$. This also shows that $a \neq 0$, for then $a^2 = 0 = f(a)$ contradicting the choice of a. We further observe that $f(x) = 0$ or $f(x) = x^2$ for any x. In any case $f(0) = 0$. Taking $x = 0$ in (1), we get

$$f(y) = f(-y).$$

Putting $x = a$ and replacing y by $-y$, we also see that

$$f(a^2 + y) = f(-y) = f(y).$$

Thus f is periodic with period a^2. This implies that

$$f(f(x)) = f(f(x) + a^2) = f(x^2 - a^2) + 4f(x)a^2.$$

Putting $y = 0$ in (1), we get another expression $f(f(x)) = f(x^2)$. Invoking the periodicity of f, we conclude that $f(x)a^2 = 0$. However, we have observed that $a \neq 0$ by our choice of a. It follows that if $f(x) \neq x^2$, then we must have $f(x) \equiv 0$. This completes our claim and determines all the solutions of the problem. ■

Problem 3.10 Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(f(x - y)) = f(x) - f(y) + f(x)f(y) - xy, \qquad (1)$$

holds good for all reals x, y.

Solution: Let us denote $f(0)$ by c. Taking $y = 0$ in (1), we obtain

$$f(f(x)) = f(x) - c + cf(x). \qquad (2)$$

Taking $x = y$ in (1), we get

$$f(c) = f(x)^2 - x^2. \qquad (3)$$

Taking $x = 0$ in (1), we obtain

$$f(f(-y)) = c - f(y) + cf(y). \qquad (4)$$

Replacing y in (4) by $-x$ and comparing the resulting expression with (2), we see that

$$2c = f(x) + f(-x) + c\big(f(x) - f(-x)\big). \tag{5}$$

However (3) shows that $f(-x)^2 - (-x)^2 = f(c) = f(x)^2 - x^2$. Thus $f(x)^2 = f(-x)^2$ giving $f(-x) = f(x)$ or $f(-x) = -f(x)$. If $f(-x_0) = f(x_0)$, for some x_0, then (5) shows that $f(x_0) = c$. Now (2) implies that $f(c) = c^2$. But then (3) gives $c^2 = f(c) = f(x_0)^2 - x_0^2 = c^2 - x_0^2$, forcing $x_0 = 0$. This leads to the conclusion $f(-x) \neq f(x)$ for all $x \neq 0$. Thus it follows that $f(-x) = -f(x)$ for all $x \neq 0$. Now for all such x, (5) reduces to $c = cf(x)$. Since we cannot have $f(x) = 1$ for all $x \neq 0$, the only possibility is $c = 0$.

We get from (3), $f(x)^2 = x^2$ giving $f(x) = x$ or $f(x) = -x$. But (2) shows that $f\big(f(x)\big) = f(x)$ and hence we can rule out $f(x) = -x$. Thus the only function which is a possible solution of (1) is $f(x) = x$ for all $x \in \mathbb{R}$. It is easy to check that $f(x) = x$ is indeed a solution. ∎

Problem 3.11 Find all functions $f : \mathbb{R}_0 \to \mathbb{R}_0$ satisfying the functional relation

$$f\big(f(x) - x\big) = 2x, \ \forall x \in \mathbb{R}_0. \tag{1}$$

Solution: (Abhay Kumar Jha) We begin with the observation that $f(x) \geq 0$ for all $x \in \mathbb{R}_0$. Define a new function g on $\mathbb{R}_0$ by setting $g(x) = f(x) - x$. Since $f(x) \geq x$, we see that $g\big(g(x)\big)$ is meaningful. A simple computation gives

$$\begin{aligned} g\big(g(x)\big) = f\big(g(x)\big) - g(x) &= f\big(f(x) - x\big) - \big(f(x) - x\big) \\ &= 3x - f(x) = 2x - g(x). \end{aligned}$$

Thus we have to find $g : \mathbb{R}_0 \to \mathbb{R}_0$ satisfying

$$g\big(g(x)\big) + g(x) = 2x. \tag{2}$$

Fix some a in $\mathbb{R}_0$ and define $u_n = g^n(a)$, where $g^n(x) = g(g^{n-1}(x))$. Then $\langle u_n \rangle$ satisfies the recurrence relation $u_{n+2} + u_{n+1} - 2u_n = 0$ for all $n \geq 0$, where $u_0 = a$. This difference equation for u_n has auxiliary equation $x^2 + x - 2 = 0$, which has solutions $x = 1$ and $x = -2$. Using the theory of difference equations, the general solution of the difference equation is given by

$$u_n = A(1)^n + B(-2)^n$$

for some constants A and B. However, we see that $g(x) \geq 0$ for all $x \in \mathbb{R}_0$, and hence $u_n = g^n(a) \geq 0$. Since $(-2)^n$ alters sign as n runs through natural numbers, we conclude that $B = 0$. This forces $u_n = A$ for all n and using the initial condition, we see that $2A = u_2 + u_1 = 2a$. we thus obtain $A = a$ and hence $u_n = a$ for all n. But then $g(a) = u_1 = a$ and we conclude that $f(a) = 2a$. Since this is true for every $a \in \mathbb{R}_0$, we arrive at the solution $f(x) = 2x$.

It is easy to verify that it is indeed a solution. ■

If we restrict our attention to functional equations on $\mathbb{Q}$, the set of all rational numbers, there are two useful ways of analysing the problems. The set of all rational numbers has, like $\mathbb{R}$, the field structure on it and it can be of significant help while solving problems on $\mathbb{Q}$. On the other hand, $\mathbb{Q}$ is also a countable set in the sense that we can set up a bijection between $\mathbb{Q}$ and $\mathbb{N}$. Hence some form of induction is also applicable here. We consider problems which effectively illustrate these ideas.

Problem 3.12 Find all functions $f : \mathbb{Q}^+ \to \mathbb{Q}^+$ satisfying the relations

(a) $f(x+1) = f(x) + 1$, for all $x \in \mathbb{Q}^+$;

(b) $f(x^3) = f(x)^3$, for all $x \in \mathbb{Q}^+$.

Solution: Putting $x = 1$ in (b), we see that $f(1)^3 = f(1)$. Since $f(1) \in \mathbb{Q}^+$, only possibility is $f(1) = 1$. This information with (a) implies that $f(2) = 2$. Now an easy induction using (a) gives $f(n) = n$ for all natural numbers n. We also observe that an induction using (a) gives us $f(r+n) = f(r) + n$ for all natural numbers n and positive rationals r.

Take any $r \in \mathbb{Q}^+$, say $r = p/q$, where p and q are natural numbers. We know that

$$\left(\frac{p}{q} + q^2\right)^3 = \left(\frac{p}{q}\right)^3 + 3p^2 + 3pq^3 + q^6.$$

Using (a) and the fact that $f(n) = n$ for all natural numbers n, we obtain

$$f\left(\left(\frac{p}{q} + q^2\right)^3\right) = f\left(\left(\frac{p}{q}\right)^3\right) + 3p^2 + 3pq^3 + q^6.$$

But (b) gives

$$\begin{aligned} f\left(\left(\frac{p}{q} + q^2\right)^3\right) &= \left(f\left(\frac{p}{q} + q^2\right)\right)^3 \\ &= \left(f\left(\frac{p}{q}\right) + q^2\right)^3 \\ &= \left(f\left(\frac{p}{q}\right)\right)^3 + \left(f\left(\frac{p}{q}\right)\right)^2 q^2 \\ &\qquad + 3f\left(\frac{p}{q}\right) q^4 + q^6. \end{aligned}$$

Comparison of these two expressions gives

$$p^2 + pq^3 = q^2\left(f\left(\frac{p}{q}\right)\right)^2 + q^4 f\left(\frac{p}{q}\right).$$

This can be written in the form

$$\left(\frac{p}{q}\right)^2 - \left(f\left(\frac{p}{q}\right)\right)^2 = q^2\left(f\left(\frac{p}{q}\right) - \frac{p}{q}\right).$$

Thus we obtain

$$r^2 - f(r)^2 = q^2\big(f(r) - r\big).$$

Suppose $f(r) \neq r$. The above relation shows that

$$-\big(r + f(r)\big) = q^2. \tag{1}$$

But r and $f(r)$ are positive rationals and q^2 is also positive. Hence the relation (1) is impossible. We conclude that $f(r) = r$. Thus f is the identity function on $\mathbb{Q}^+$. ∎

Problem 3.13 Find all $f : \mathbb{Q} \setminus \{0\} \to \mathbb{Q} \setminus \{0\}$ such that

$$f\left(\frac{x+y}{3}\right) = \frac{f(x) + f(y)}{2}, \tag{1}$$

for all $x, y \in \mathbb{Q} \setminus \{0\}$.

Solution: Taking $x = z$ and $y = 2z$ in (1), we obtain

$$f(z) = \frac{f(z) + f(2z)}{2},$$

which shows that $f(2z) = f(z)$ for all $z \in \mathbb{Q} \setminus \{0\}$. Similarly taking $x = y = 3z$ in (1), we obtain

$$f(3z) = f(2z) = f(z), \quad \text{for all } z \in \mathbb{Q} \setminus \{0\}. \tag{2}$$

Suppose we know that $f(kz) = f(z)$ for all $k \leq n$. Taking $x = 3nz$ and $y = 3z$ in (1), we obtain

$$f\big((n+1)z\big) = \frac{f(3nz) + f(3z)}{2}.$$

Since $f(3nz) = f(nz) = f(z)$ and $f(3z) = f(z)$ by induction hypothesis, we obtain

$$f\big((n+1)z\big) = f(z).$$

It follows that $f(nz) = f(z)$ for all natural numbers n and $z \in \mathbb{Q} \setminus \{0\}$. Taking $z = 1$, we see that $f(n) = f(1)$ for all natural numbers.

Let p/q be a positive rational number. Then we have

$$f(1) = f(p) = f\left(q\left(\frac{p}{q}\right)\right) = f\left(\frac{p}{q}\right).$$

It follows that $f(r) = f(1)$ for all positive rationals r.

Taking $x = 6$ and $y = -3$ in (1), we get

$$f(1) = \frac{f(6) + f(-3)}{2},$$

showing that $f(-3) = f(1)$. But (2) implies that $f(-3) = f(-1)$. Thus we obtain $f(-1) = f(1)$. An entirely similar analysis gives $f(r) = f(-1)$ for all negative rationals r. Since $f(1) = f(-1)$, we conclude that $f(r) = f(1)$ for all rationals $r \neq 0$. Thus f is a constant function on $\mathbb{Q} \setminus \{0\}$. ∎

Problem 3.14 Find all functions $f : \mathbb{Q} \to \mathbb{Q}$ such that

$$f(x+y) + f(x-y) = 2f(x) + 2f(y), \tag{1}$$

for all rationals x, y.

(Nordic Contest-1998)

Solution: If we set $x = y = 0$ in (1), we see that $f(0) = 0$ and hence $f(2x) = 4f(x)$. By an easy induction, we can show that $f(nx) = n^2 f(x)$ for all natural numbers n and rational numbers x. Taking $x = 0$, we also observe that $f(-y) = f(y)$ so that $f(nx) = n^2 f(x)$ for all integers n. If $x = p/q$ is a rational, then

$$q^2 f(x) = f(qx) = f(p) = p^2 f(1).$$

We thus obtain $f(x) = cx^2$ for all rationals x, where $c = f(1)$.

■

Here is a problem which can be solved using a variant of induction principle. On the other hand it also reveals the intricacies of solving an equation.

Problem 3.15 Find all $f : \mathbb{Q}^+ \to \mathbb{Q}^+$ satisfying:

(a) $f(x) + f\left(\frac{1}{x}\right) = 1$, for all $x \in \mathbb{Q}^+$;

(b) $f(1 + 2x) = \frac{1}{2}f(x)$, for all $x \in \mathbb{Q}^+$.

Solution: Taking $x = 1$ in (a), we get $f(1) = 1/2$. If we set $x = 1/2$ in (a) and (b), we see that

$$f\left(\frac{1}{2}\right) + f(2) = 1; \; f(2) = \frac{1}{2}f\left(\frac{1}{2}\right).$$

Solving for $f(2)$ and $f\left(\frac{1}{2}\right)$, we obtain

$$f(2) = \frac{1}{3}, \; f\left(\frac{1}{2}\right) = \frac{2}{3}.$$

Taking $x = 1$ in (b), we see that

$$f(3) = \frac{1}{2}f(1) = \frac{1}{4}.$$

Now if we use (a), we compute

$$f\left(\frac{1}{3}\right) = 1 - f(3) = \frac{3}{4}.$$

Similarly substituting $x = 1/4$ in (b) we see that

$$f\left(\frac{3}{2}\right) = \frac{1}{2}f\left(\frac{1}{4}\right),$$

and taking $x = 3/2$ in (a) we obtain

$$f(4) = \frac{1}{2} f\left(\frac{3}{2}\right).$$

If we eliminate $f\left(\frac{3}{2}\right)$ from these two relations, we get

$$f(4) = \frac{1}{4} f\left(\frac{1}{4}\right).$$

But we know from (a) that

$$f(4) + f\left(\frac{1}{4}\right) = 1.$$

It follows that

$$f(4) = \frac{1}{5}, \quad \text{and} \quad f\left(\frac{1}{4}\right) = \frac{4}{5}.$$

These in turn leads to

$$f\left(\frac{3}{2}\right) = \frac{2}{5}, \quad \text{and} \quad f\left(\frac{2}{3}\right) = \frac{3}{5}.$$

An inspection of the values so far obtained reveals that $f(x) = \dfrac{1}{1+x}$ is possibly the required function. We show that this indeed is the solution of our functional equation. We adopt the following procedure to prove our claim by induction. For each rational $r = p/q \in \mathbb{Q}^+$ with $\gcd(p, q) = 1$, we define $d(r) = p + q$ which is a natural number. We show that $f(r) = \dfrac{1}{1+r}$ by using induction on $d(r)$. We have so far verified this claim for all $r \in \mathbb{Q}^+$ for which $d(r) \leq 5$ holds.

Suppose this result is true for all $r \in \mathbb{Q}^+$ such that $d(r) \leq N$. Take any $r = p/q \in \mathbb{Q}^+$ such that $\gcd(p, q) = 1$ and $d(r) = p + q = N + 1$. We have

$$f\left(\frac{q}{p}\right) = f\left(1 + 2\left(\frac{q-p}{2p}\right)\right) = \frac{1}{2} f\left(\frac{q-p}{2p}\right).$$

If q and p are both odd, then 2 divides $q-p$. Thus

$$d\left(\frac{q-p}{2p}\right) \leq \frac{q-p}{2}+p = \frac{q+p}{2} \leq N.$$

By induction hypothesis, we obtain

$$f\left(\frac{q-p}{2p}\right) = \frac{1}{1+\dfrac{q-p}{2p}} = \frac{2p}{q+p}.$$

Thus we get

$$f\left(\frac{q}{p}\right) = \frac{p}{p+q},$$

and

$$f\left(\frac{p}{q}\right) = 1 - f\left(\frac{q}{p}\right) = \frac{q}{p+q} = \frac{1}{1+\dfrac{p}{q}}.$$

Suppose p and q have different parity. If $q-p>2p$, then

$$f\left(\frac{q-p}{2p}\right) = f\left(1+2\left(\frac{q-3p}{4p}\right)\right) = \frac{1}{2}f\left(\frac{q-3p}{4p}\right),$$

and hence

$$f\left(\frac{q}{p}\right) = \frac{1}{2}f\left(\frac{q-p}{2p}\right) = \frac{1}{2^2}f\left(\frac{q-(2^2-1)p}{2^2p}\right).$$

Let s_1 be the least positive integer such that $2^{s_1}p > q-(2^{s_1}-1)p$. Using the above procedure we arrive at

$$f\left(\frac{q}{p}\right) = \frac{1}{2^{s_1}}f\left(\frac{q-(2^{s_1}-1)p}{2^{s_1}p}\right).$$

Put $q_1 = 2^{s_1}p$ and $p_1 = q-(2^{s_1}-1)p$. Then we can express $f(q_1/p_1)$ by

$$f\left(\frac{q_1}{p_1}\right) = 1 - f\left(\frac{p_1}{q_1}\right) = 1 - 2^{s_1}f\left(\frac{q}{p}\right).$$

We observe that

$$d\left(\frac{q_1}{p_1}\right) = q_1 + p_1 = q + p = N + 1,$$

and $\gcd(p_1, q_1) = \gcd(p, q) = 1$. Let s_2 be the least positive integer such that

$$2^{s_2} p_1 > q_1 - (2^{s_2} - 1)p_1.$$

Then we obtain

$$f\left(\frac{q_1}{p_1}\right) = \frac{1}{2^{s_2}} f\left(\frac{p_2}{q_2}\right),$$

where $q_2 = 2^{s_2} p_1$ and $p_2 = q_1 - (2^{s_2} - 1)p_1$. Thus we obtain

$$\begin{aligned} f\left(\frac{q_2}{p_2}\right) &= 1 - f\left(\frac{p_2}{q_2}\right) \\ &= 1 - 2^{s_2} f\left(\frac{q_1}{p_1}\right) \\ &= 1 - 2^{s_2}\left(1 - f\left(\frac{p_1}{q_1}\right)\right) \\ &= 1 - 2^{s_2} + 2^{s_2+s_1} f\left(\frac{q}{p}\right). \end{aligned}$$

Continuing this process, we get a sequence $\langle (p_k, q_k) \rangle$ such that

1. $\gcd(p_k, q_k) = 1$ for all k;
2. $p_k + q_k = p + q = N + 1$, for all k;
3. $p_k = q_{k-1} - (2^{s_k} - 1)p_{k-1}$ and $q_k = 2^{s_k} p_{k-1}$(here $p_0 = p$ and $q_0 = q$) where s_k is the least positive integer such that $2^{s_k} p_{k-1} > q_{k-1} - (2^{s_k} - 1)p_{k-1}$;
4. $2^{s_k} f\left(\frac{q_{k-1}}{p_{k-1}}\right) = f\left(\frac{p_k}{q_k}\right)$.

Now there are only finitely many solutions to the equation $a+b=N+1$ with $\gcd(a,b)=1$. Hence there must be repetitions in the in the sequence $\langle(p_k,q_k)\rangle$. Let us suppose

$$(p_m,q_m)=(p_{m+t},q_{m+t}).$$

For convenience, let us also introduce

$$2^{s_{m+t}}+s_{m+t-1}+\cdots+s_{m+t-r}=u_r.$$

We then obtain

$$\begin{aligned}
f\left(\frac{p_{m+t}}{q_{m+t}}\right) &= 2^{s_{m+t}}f\left(\frac{q_{m+t-1}}{p_{m+t-1}}\right)\\
&= 2^{s_{m+t}}-2^{s_{m+t}}f\left(\frac{p_{m+t-1}}{q_{m+t-1}}\right)\\
&= u_0-u_1+u_1f\left(\frac{p_{m+t-2}}{q_{m+t-2}}\right)\\
&\vdots\\
&= u_0-u_1+\cdots+(-1)^{t-1}u_{t-1}\\
&\qquad +(-1)^tu_{t-1}f\left(\frac{p_m}{q_m}\right).
\end{aligned}$$

Now using $p_{m+t}=p_m$ and $q_{m+t}=q_m$ we solve for $f\left(\frac{p_m}{q_m}\right)$:

$$f\left(\frac{p_m}{q_m}\right)=\frac{u_0-u_1+\cdots+(-1)^{t-1}u_{t-1}}{1-(-1)^tu_t}.$$

However, we also have

$$\begin{aligned}
p_{m+t} &= q_{m+t-1}-\left(2^{s_{m+t}}-1\right)p_{m+t-1}\\
&= q_{m+t-1}+p_{m+t-1}-2^{s_{m+t}}p_{m+t-1}\\
&= (p_m+q_m)-2^{s_{m+t}}p_{m+t-1}.
\end{aligned}$$

An easy induction gives

$$\begin{aligned}
p_{m+t} &= (p_m+q_m)\{1-u_0+u_1-u_2+\cdots\\
&\qquad +(-1)^{t-1}u_{t-2}\}+(-1)^tu_{t-1}p_m.
\end{aligned}$$

Using $p_{m+t} = p_m$, we obtain

$$\frac{p_m}{p_m + q_m} = \frac{1 - u_0 + u_1 - u_2 + \cdots + (-1)^{t-1} u_{t-2}}{1 - (-1)^t u_{t-1}}.$$

But we also note that

$$\begin{aligned} f\left(\frac{q_m}{p_m}\right) &= 1 - f\left(\frac{p_m}{q_m}\right) \\ &= 1 - \left\{\frac{u_0 - u_1 + u_2 - \cdots + (-1)^{t-1} u_{t-1}}{1 - (-1)^t u_{t-1}}\right\} \\ &= \frac{1 - u_0 + u_1 - u_2 + \cdots + (-1)^{t-1} u_{t-2}}{1 - (-1)^t u_{t-1}} \\ &= \frac{p_m}{p_m + q_m}. \end{aligned}$$

Thus it follows that

$$f\left(\frac{p_m}{q_m}\right) = 1 - f\left(\frac{q_m}{p_m}\right) = \frac{q_m}{p_m + q_m}.$$

On the other hand we also observe that

$$f\left(\frac{p_m}{q_m}\right) = 2^{s_m} f\left(\frac{q_{m-1}}{p_{m-1}}\right)$$

so that

$$\begin{aligned} f\left(\frac{q_{m-1}}{p_{m-1}}\right) &= \frac{1}{2^{s_m}} f\left(\frac{p_m}{q_m}\right) \\ &= \left(\frac{1}{2^{s_m}}\right)\left(\frac{q_m}{p_m + q_m}\right) \\ &= \frac{p_{m-1}}{p_m + q_m} \\ &= \frac{p_{m-1}}{p_{m-1} + q_{m-1}}. \end{aligned}$$

This gives

$$f\left(\frac{p_{m-1}}{q_{m-1}}\right) = 1 - f\left(\frac{q_{m-1}}{p_{m-1}}\right) = \frac{q_{m-1}}{p_{m-1} + q_{m-1}}.$$

Continuing this process by induction , we arrive at

$$f\left(\frac{p}{q}\right) = f\left(\frac{p_0}{q_0}\right) = \frac{q_0}{p_0 + q_0} = \frac{q}{p+q}.$$

Thus we finally obtain

$$f\left(\frac{p}{q}\right) = \frac{q}{p+q} = \frac{1}{1+\dfrac{p}{q}},$$

showing that $f(r) = \dfrac{1}{1+r}$ for all positive rationals r. ■

Problem 3.16 Find a function $f : \mathbb{Q}^+ \to \mathbb{Q}^+$ such that

$$f\big(xf(y)\big) = \frac{f(x)}{y}, \quad \text{for all } x, y \in \mathbb{Q}^+. \tag{1}$$

(IMO-1990)

Solution: Taking $x = 1$ in (1), we obtain

$$f\big(f(y)\big) = \frac{f(1)}{y}, \quad \text{for all } y \in \mathbb{Q}^+. \tag{2}$$

Since $f(x)$ lies in $\mathbb{Q}^+$ for all $x \in \mathbb{Q}^+$, it follows that $f(1) \neq 0$. This implies that f is an one-one function. Taking $y = 1$ in (2), we also get $f\big(f(1)\big) = f(1)$ and the injectivity of f shows that $f(1) = 1$. Using this in (2), we obtain

$$f\big(f(y)\big) = \frac{1}{y}, \quad \text{for all } y \in \mathbb{Q}^+. \tag{3}$$

Since $1/y$ varies over $\mathbb{Q}^+$ as y takes values in $\mathbb{Q}^+$, we conclude that f is also on-to. Thus f is a bijection on $\mathbb{Q}^+$.

We show that f is also a multiplicative function; $f(xy) = f(x)f(y)$ for all $x, y \in \mathbb{Q}^+$. Take x, y in $\mathbb{Q}^+$, and choose $t \in$

$\mathbb{Q}^+$ such that $y = f(t)$. This is possible by the surjectivity of f. Thus we get

$$f(xy) = f\big(xf(t)\big) = \frac{f(x)}{t} = f(x)f\big(f(t)\big) = f(x)f(y).$$

Observe that we have used (3) here. We also obtain

$$f\left(\frac{1}{x}\right) = f\big(f\big(f(x)\big)\big) = \frac{1}{f(x)}.$$

We define f on the set of all primes such that $f\big(f(p)\big) = 1/p$ and then extend it to all positive rationals using reciprocity and multiplicativity. This can be done in infinitely many ways. Let $\{A, B\}$ be a partition of the set of all primes such that A and B are both infinite. For example, we can choose

$$A = \{2\} \cup \{p : p \equiv 1 \pmod 4\},\ B = \{p : p \equiv 3 \pmod 4\}.$$

Both are infinite and can be enumerated as $A = \langle p_n \rangle$ and $B = \langle q_n \rangle$. Define $f(1) = 1$, $f(p_j) = q_j$ and $f(q_j) = 1/p_j$, for all $j \geq 1$. If

$$n = a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_k^{\alpha_k}$$

is the prime decomposition of n, we set

$$f(n) = f(a_1)^{\alpha_1} f(a_2)^{\alpha_2} \cdots f(a_k)^{\alpha_k}.$$

Finally, if $r \in \mathbb{Q}^+$ is such that $r = m/n$ for some $m, n \in \mathbb{N}$, then we set $f(r) = f(m)/f(n)$. It is easy to verify that such a function satisfies all the requirement of the given problem. ∎

Problem 3.17 Find all functions $f : \mathbb{Q} \to \mathbb{Q}$ such that

$$f(xy) = f(x)f(y) - f(x+y) + 1, \tag{1}$$

for all rationals x, y.

Solution: Putting $x = y = 0$ in (1), we see that $\big(f(0) - 1\big)^2 = 0$ forcing $f(0) = 1$. Setting $x = 1$ and $y = -1$ in (1), we also obtain $f(-1) = f(1)f(-1)$. This leads to two possibilities: $f(-1) = 0$ or $f(1) = 1$.

Case 1. Suppose $f(-1) = 0$. Replacing y by yz in (1), we get

$$\begin{aligned} f(xyz) &= f(x)f(yz) - f(x + yz) + 1 \\ &= f(x)[f(y)f(z) - f(y + z) + 1] - f(x + yz) + 1. \end{aligned}$$

On the other hand, we can also express $f(xyz)$ in the form

$$\begin{aligned} f(xyz) &= f(xy)f(z) - f(xy + z) + 1 \\ &= \big[f(x)f(y) - f(x + y) + 1\big]f(z) \\ &\qquad - f(xy + z) + 1. \end{aligned}$$

Comparing these two expressions, we see that

$$\begin{aligned} f(x)f(y + z) - f(x) + f(x + yz) \\ = f(z)f(x + y) - f(z) + f(xy + z). \end{aligned} \tag{2}$$

Now taking $z = -1$ in (2) and using $f(-1) = 0$, we obtain

$$f(x)f(y - 1) - f(x) + f(x - y) = f(xy - 1). \tag{3}$$

If we set $x = 1$ in (3) we get

$$f(y - 1)\{1 - f(1)\} = f(1 - y) - f(1).$$

Replacing y by $y + 1$, we can write the above expression in the form

$$f(y)\{1 - f(1)\} = f(-y) - f(1), \tag{4}$$

for all rationals y. Taking $y = 1$ in this relation, we get $f(1)\big[2 - f(1)\big] = 0$. Thus we have $f(1) = 0$ or $f(1) = 2$.

A. If $f(1) = 0$, then (4) implies that $f(y) = f(-y)$. Replacing y by $-y$ in (1), we get

$$f(xy) = f(x)f(y) - f(x-y) + 1. \tag{5}$$

Subtracting (1) from (5), we are lead to the relation

$$f(x+y) - f(x-y) = 0,$$

for all rationals x, y. Taking $y = x$, we obtain $f(2x) = f(0) = 1$. It follows that $f(x) = 1$ for all rationals x. But this contradicts $f(1) = 0$. We conclude that there is no solution in the case $f(1) = 0$.

B. Suppose $f(1) = 2$, so that (4) takes the form

$$1 - f(y) = f(-y) - 1.$$

Introducing $g(x) = 1 - f(x)$, we see that $g(-y) = -g(y)$ showing that g is an *odd* function. Now (1) can be written as

$$g(xy) = g(x) + g(y) - g(x)g(y) - g(x+y). \tag{6}$$

Replacing y by $-y$ and using that g is *odd*, we also get

$$-g(xy) = g(x) - g(y) + g(x)g(y) - g(x-y). \tag{7}$$

Adding (6) and (7), we obtain the relation

$$g(x+y) + g(x-y) = 2g(x).$$

Thus $g(2x) = 2g(x)$. We can prove by induction that $g(nx) = ng(x)$ for all natural numbers n and this can be extended to all integers using that g is an *odd* function. In particular $g(n) = ng(1)$ for all $n \in \mathbb{Z}$. From this we can conclude that $g(x) = xg(1)$ for all rationals x. But $g(1) = 1 - f(1) = -1$. It follows that $g(x) = -x$ for all $x \in \mathbb{Q}$ and hence $f(x) = 1 + x$ for all rationals x.

Case 2. In the case $f(1) = 1$, we set $z = 1$ in (2) to get

$$f(xy+1) - f(x)f(y+1) + f(x) = 1.$$

If we take $y = -1$ here, we obtain $f(1-x) = 1$ for all $x \in \mathbb{Q}$. We conclude that $f(x) \equiv 1$.

Thus there are two solutions to the problem: $f(x) \equiv 1$ and $f(x) = 1 + x$ for all $x \in \mathbb{Q}$. It is easily checked that these functions are indeed solutions to the given functional equation.

■

We have seen earlier that the concept and properties of fixed points are often helpful in solving certain functional equations on natural numbers. This is also true of functional equations on $\mathbb{R}$. We illustrate this with a problem.

Problem 3.18 Find all functions $f : (-1, \infty) \to (-1, \infty)$ such that

(a) $f\big(x + f(y) + xf(y)\big) = y + f(x) + yf(x)$, for all $x, y \in (-1, \infty)$;

(b) $\dfrac{f(x)}{x}$ is strictly increasing on each of the intervals $(-1, 0)$ and $(0, \infty)$.

(IMO-1994)

Solution: Let $f : (-1, \infty) \to (-1, \infty)$ be a function of the desired type. Since $\dfrac{f(x)}{x}$ is strictly increasing on the interval $(-1, 0)$, the equation $f(x) = x$ can have at most one solution in $(-1, 0)$. Similarly, $f(x) = x$ can have at most one solution in $(0, \infty)$. Moreover $x = 0$ may be a solution of $f(x) = x$. Thus the equation $f(x) = x$ can have at most three solutions in $(-1, \infty)$. In other words, there are at most three fixed points of $f(x)$ in the domain $(-1, \infty)$.

Suppose $u \in (-1,0)$ is a fixed point of $f(x)$. Thus we have $f(u) = u$. Taking $x = y = u$ in (a), we see that $f(2u+u^2) = 2u+u^2$. This shows that $2u+u^2$ is also a fixed point of $f(x)$. We claim that $2u+u^2$ is in the interval $(-1,0)$. In fact $2u+u^2 = u(2+u) < 0$, since $u < 0$ and $2+u > 1 > 0$ because $u > -1$. On the other hand $2u + u^2 > -1$ because $2u + u^2 + 1 = (u+1)^2 > 0$. Since there can be at most one fixed point of $f(x)$ in $(-1,0)$, we conclude that $2u+u^2 = u$. This forces $u(u+1) = 0$ contradicting the assumption that $u \in (-1,0)$. It follows that there is no fixed point of $f(x)$ in $(-1,0)$. Similar analysis shows that $f(x)$ has no fixed point in $(0,\infty)$ as well. Thus 0 is the only possible fixed point of $f(x)$ if at all it has any. However taking $x = y$ in (i), we see that $f\big(x + f(x) + xf(x)\big) = x + f(x) + xf(x)$ for all $x \in (-1,\infty)$. Thus each $x + f(x) + xf(x)$, $x \in (-1,\infty)$, is a fixed point of f. We conclude that $x + f(x) + xf(x) = 0$ for all $x \in (-1,\infty)$. This leads to

$$f(x) = -\left(\frac{x}{1+x}\right), \quad \text{for all } x \in (-1,\infty).$$

We see that

$$x + f(y) + xf(y) = x - \frac{y}{1+y} - \frac{xy}{1+y} = \frac{x-y}{1+y}.$$

Thus we obtain

$$f\big(x + f(y) + xf(y)\big) = f\left(\frac{x-y}{1+y}\right) = \frac{y-x}{1+y}.$$

On the other hand we see that

$$y + f(x) + yf(x) = \frac{y-x}{1+y}.$$

It follows that that $f(x) = -\left(\dfrac{x}{1+x}\right)$ indeed satisfies (a). We can easily check that it also satisfies (b). ∎

The fixed points can also be used in proving non-existence of solutions to some functional equations. The following problem illustrates this point.

Problem 3.19 Does there exist a function $f : \mathbb{R} \to \mathbb{R}$ such that $f\big(f(x)\big) = x^2 - 2$ for all real x?

Solution: No. In fact, we prove a more general non-existence theorem.

Suppose X is a set, $g : X \to X$ has exactly two fixed points $\{a, b\}$ and $g \circ g$ has exactly four fixed points $\{a, b, c, d\}$. Then there is no function $f : X \to X$ such that $g = f \circ f$.

We first prove that $g(c) = d$ and $g(d) = c$. Suppose $g(c) = y$. Then $c = g\big(g(c)\big) = g(y)$, and hence $g\big(g(y)\big) = g(c) = y$. Thus y is a fixed point of $g \circ g$. If $y = a$, then we see that $a = g(a) = g(y) = c$ leading to a contradiction. Similarly $y = b$ forces $b = c$. If $y = c$, then $c = g(y) = g(c)$ so that c is one of a, b. Thus the only possibility is $y = d$ giving $g(c) = d$. A similar analysis gives $g(d) = c$.

Suppose there exists a function $f : X \to X$ such that $g(x) = f\big(f(x)\big)$ for all $x \in X$. Then it is easy to see that $f\big(g(x)\big) = g\big(f(x)\big)$ for all $x \in X$. Let $x_0 \in \{a, b\}$. Then $f(x_0) = f\big(g(x_0)\big) = g\big(f(x_0)\big)$, so that $f(x_0)$ is a fixed point of g. Hence $f(x_0) \in \{a, b\}$. Similarly, it is easy to show that $x_1 \in \{a, b, c, d\}$ implies that $f(x_1) \in \{a, b, c, d\}$.

Consider $f(c)$. This lies in $\{a, b, c, d\}$. If $f(c) = a$, then $f(a) = f\big(f(c)\big) = g(c) = d$, a contradiction since f maps $\{a, b\}$ in to it self. Similarly, $f(c) = b$ gives $f(b) = d$, which is impossible. If $f(c) = c$, then $c = f(c) = f\big(f(c)\big) = g(c) = d$ from our earlier observation. This contradicts the distinctness of c and d. If $f(c) = d$, then $f(d) = f\big(f(c)\big) = g(c) = d$ and this gives $g(d) = f\big(f(d)\big) = f(d) = d$ contradicting our observation that $g(d) = c$. Thus $f(c)$ cannot be an element of $\{a, b, c, d\}$.

We conclude that there is no function $f : X \to X$ such that $g = f \circ f$.

We now use the above result to show that there is no function $f : \mathbb{R} \to \mathbb{R}$ such that $f\big(f(x)\big) = x^2 - 2$. Consider $g(x) = x^2 - 2$. It has two fixed points $2, -1$ and $g \circ g$ has four fixed points $\dfrac{-1+\sqrt{5}}{2}, \dfrac{-1-\sqrt{5}}{2}, 2, -1$. Hence there is no function f such that $g = f \circ f$ and this proves our assertion.

Remark: In an article [5], R. E. Rice, B. Schweizer and A. Sklar show that there is no function $f : \mathbb{C} \to \mathbb{C}$ such that $f\big(f(z)\big) = P(z)$, where $P(z)$ is a polynomial of degree 2.

■

Here is a functional equation which uses a totally different idea. The function is first characterised on its range and this information is used to get the complete description.

Problem 3.20 Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f\big(x - f(y)\big) = f\big(f(y)\big) + xf(y) + f(x) - 1, \qquad (1)$$

holds for all $x, y \in \mathbb{R}$.

(IMO-1999)

Solution: We easily see that $f(x) = 1 - \dfrac{x^2}{2}$ satisfies the equation (1). We show that this is the only function which obey the relation (1). Let S denote the range of f. Put $c = f(0)$. Taking $x = y = 0$ in (1), we obtain

$$f(-c) = f(c) + c - 1.$$

This shows that $c \neq 0$. Taking $x = f(y)$ in (1), we also get

$$c = f(x) + x^2 + f(x) - 1.$$

This gives

$$f(x) = \frac{c+1}{2} - x^2, \qquad (2)$$

whenever $x = f(y)$. This determines f on S, the range of f. Taking $y = 0$ in (1), we get

$$f(x-c) = f(c) + cx + f(x) - 1.$$

This can be written in the form $f(x-c)-f(x) = cx+f(c)-1$. Consider the set $\{cx + f(c) - 1 : x \in \mathbb{R}\}$. Since $c \neq 0$, it follows that this set is $\mathbb{R}$ itself. Thus we conclude that

$$\{f(x-c) - f(x) : x \in \mathbb{R}\} = \mathbb{R}.$$

We use this to determine f on $\mathbb{R}$.

Fix any $x \in \mathbb{R}$. We can find $y_1, y_2 \in S$ such that $x = y_1 - y_2$. Let $y_2 = f(x_2)$. Then

$$\begin{aligned} f(x) = f\big(y_1 - f(x_2)\big) &= f\big(f(x_2)\big) + y_1 f(x_2) + f(y_1) - 1 \\ &= f(y_2) + y_1 y_2 + f(y_1) - 1. \end{aligned}$$

But we know f on S: from (2) we see that

$$f(y_2) = \frac{c+1}{2} - \frac{y_2^2}{2}, \quad f(y_1) = \frac{c+1}{2} - \frac{y_1^2}{2}.$$

Putting these values, we obtain

$$f(x) = c - \frac{(y_2 - y_1)^2}{2} = c - \frac{x^2}{2}.$$

This is valid for all real x. If we take x in S, then we also know that

$$f(x) = \frac{c+1}{2} - \frac{x^2}{2}.$$

Comparing these expressions, we conclude that $c = 1$. Thus we obtain

$$f(x) = 1 - \frac{x^2}{2}, \quad \text{for all } x \in \mathbb{R}.$$

Alternate Solution: As in the first solution, we take $c = f(0)$. Putting $x = f(y)$ in (1), we can solve for $f\big(f(y)\big)$:

$$f\big(f(y)\big) = \frac{c+1-f(y)^2}{2}. \tag{3}$$

Introducing $g(x) = f(x) + \frac{x^2}{2}$, it is easy to compute

$$g\big(x - f(y)\big) = g(x) + \frac{c-1}{2}. \tag{4}$$

Note that the given equation has no constant solution. Thus we may find y_0 such that $f(y_0) \neq 0$. Taking $x = 1/f(y_0)$ and $y = y_0$ in (1), we obtain

$$f\big(x - f(y_0)\big) = f\big(f(y_0)\big) + f(x).$$

Setting $x - f(y_0) = a$, and $f(y_0) = b$, we obtain $f(a) = f(b) + f(x)$. Thus (4) gives

$$\begin{aligned} g(x) + \frac{c-1}{2} &= g\big(x - f(a)\big) = g\big(x - f(b) - f(x)\big) \\ &= g\big(x - f(b)\big) + \frac{c-1}{2} = g(x) + c - 1. \end{aligned}$$

It follows that $c = 1$ and now (4) shows that

$$g\big(x - f(y)\big) = g(x), \tag{5}$$

for all reals x, y. Thus we obtain that every element in the range of f is a period for g. However putting $f(0) = c = 1$ in (3), we obtain $f(1) = f\big(f(0)\big) = 1/2$. Also taking $y = 0$ in (1), we see that $f(x-1) = x + f(x) - \frac{1}{2}$. We have proved that $\frac{1}{2}$, $f(x)$ and $x + f(x) - \frac{1}{2}$ are periods of g. Since a linear combination of several periods is again a period, x it self is period for g. Since this is true for every real number x, we conclude that g is a constant function. However $g(0) = f(0) = 1$ and we get $g(x) \equiv 1$. The definition of g shows that $f(x) = 1 - \frac{x^2}{2}$. ∎

Some of functional equations may require a single or a combination of several ideas in their solutions. This is illustrated in the solution of the following few problems.

Problem 3.21 Suppose f, g, h are functions from $\mathbb{R}$ to $\mathbb{R}$ such that

$$\frac{f(x)-g(y)}{x-y} = h\left(\frac{x+y}{2}\right), \quad \text{for all } x, y \in \mathbb{R}, x \neq y. \quad (1)$$

Prove that

$$f(x) = g(x) = ax^2 + bx + c, \; h(x) = 2ax + b,$$

for all $x \in \mathbb{R}$, where a, b, c are some constants.

Solution: Replacing x by $x+y$ and y by $x-y$ in (1), we get

$$\frac{f(x+y)-g(x-y)}{2y} = h(x), \quad \text{for all } x, y \in \mathbb{R}, y \neq 0. \quad (2)$$

Now we replace y by $-y$ in (2) to obtain

$$\frac{f(x-y)-g(x+y)}{-2y} = h(x), \quad \text{for all } x, y \in \mathbb{R}, y \neq 0. \quad (3)$$

Taking $x = u+v$ in (2) and $x = u-v$ in (3) and adding the resulting expressions, we obtain

$$\begin{aligned} h(u+v)+h(u-v) = & \frac{1}{2y}\Big\{f(u+v+y)-g(u+v-y) \\ & + f(u-v+y)-g(u-v-y)\Big\} \\ = & \frac{1}{2y}\Big\{f\big(u+[v+y]\big)-g\big(u-[v+y]\big)\Big\} \\ & + \frac{1}{2y}\Big\{f\big(u-[v-y]\big)-g\big(u+[v-y]\big)\Big\} \\ = & \frac{1}{2y}\Big\{2(v+y)h(u)-2(v-y)h(u)\Big\}. \end{aligned}$$

Here we have used again (2) and (3). Thus we obtain

$$h(u+v)+h(u-v) = 2h(u), \quad \text{for all } u, v \in \mathbb{R}. \quad (4)$$

Putting $u+v=s$, $u-v=t$ in the above expression, we get

$$h\left(\frac{s+t}{2}\right)=\frac{h(s)+h(t)}{2}, \quad \text{for all } s,t\in\mathbb{R}. \tag{5}$$

Define a function $F:\mathbb{R}\to\mathbb{R}$, by setting

$$F(s)=h(s)-h(0), \quad s\in\mathbb{R}.$$

Then we obtain from (5)

$$\begin{aligned} F(s)+F(t) &= h(s)+h(t)-2h(0) \\ &= 2\left\{\frac{h(s)+h(t)}{2}-h(0)\right\} \\ &= 2\left\{h\left(\frac{s+t}{2}\right)-h(0)\right\} \\ &= 2F\left(\frac{s+t}{2}\right). \end{aligned}$$

We also observe that $F(0)=0$. Taking $t=0$ in the above expression, we see that

$$F(s)=2F\left(\frac{s}{2}\right), \quad \text{for all } s\in\mathbb{R}.$$

Using this back in the preceding expression, we arrive at

$$F(s+t)=F(s)+F(t), \quad \text{for all } s,t\in\mathbb{R}. \tag{6}$$

We can write (2) in the form

$$\frac{f(x+y)-g(x-y)}{2y}=h(x)=B+F(x), \tag{7}$$

where $B=h(0)$ is a constant. Putting $y=x$ and $y=-x$ successively in (7), we obtain

$$f(2x)=g(0)+2Bx+2xF(x);$$

$$g(2x) = f(0) + 2Bx + 2xF(x).$$

Replacing x by $x/2$ in these relations, we get

$$f(x) = g(0) + Bx + xF\left(\frac{x}{2}\right);$$

$$g(x) = f(0) + Bx + xF\left(\frac{x}{2}\right).$$

Substituting these back in (2), we obtain

$$\frac{1}{2y}\Big\{g(0) - f(0) + 2By + yF(x) + xF(y)\Big\} = h(x). \qquad (8)$$

We have used the additivity relation (6) for F. Taking $x = 1$ in (8), we get

$$\frac{1}{2y}\Big\{g(0) - f(0) + 2By + yF(1) + F(y)\Big\} = h(1).$$

We solve for $F(y)$ to get

$$F(y) = \big(2h(1) - 2B - F(1)\big)y + f(0) - g(0) = dy + f(0) - g(0),$$

where $d = 2h(1) - 2B - F(1)$. Putting this back in (6), we see that

$$d(x+y) + f(0) - g(0) = dx + f(0) - g(0) + dy + f(0) - g(0).$$

It follows that $f(0) - g(0) = 0$, and hence $F(y) = dy$ for all real y. Putting this back in the expressions for $f(x)$ and $g(x)$, we obtain

$$f(x) = g(0) + Bx + \left(\frac{d}{2}\right)x^2,$$

$$g(x) = f(0) + Bx + \left(\frac{d}{2}\right)x^2.$$

Since $f(0) = g(0)$, we conclude that

$$f(x) = g(x) = ax^2 + bx + c,$$

where $a = d/2$, $b = B$ and $c = f(0) = g(0)$. Finally

$$h(x) = F(x) + h(0) = dx + B = 2ax + b.$$

■

Problem 3.22 Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying $f(0) = 0$ and the functional relation

$$f\left(\frac{x^2+y^2}{2xy}\right) = \frac{f(x)^2+f(y)^2}{2f(x)f(y)}, \ \forall x, y \in \mathbb{R}, x \neq 0, y \neq 0. \tag{1}$$

Solution: We first observe that $f(x) \neq 0$ if $x \neq 0$. Putting $x = y = 1$, we see that that $f(1) = 1$. Let us take $y = xz$ in (1). We obtain

$$f\left(\frac{x^2+(xz)^2}{2x(xz)}\right) = \frac{f(x)^2+f(xz)^2}{2f(x)f(xz)}.$$

But we observe that

$$f\left(\frac{x^2+(xz)^2}{2x(xz)}\right) = f\left(\frac{1+z^2}{2z}\right) = \frac{1+f(z)^2}{2f(z)}.$$

Comparing these expressions, we get a quadratic relation in $f(zx)$:

$$f(z)f(xz)^2 - f(x)\big(1+f(z)^2\big)f(xz) + f(z)f(x)^2 = 0.$$

Solving this quadratic equation, we obtain

$$f(xz) = \frac{f(x)}{f(z)}, \quad \text{or} \quad f(xz) = f(x)f(z).$$

Suppose $f(xz) = \dfrac{f(x)}{f(z)}$. Interchanging x and z, we also get $f(zx) = \dfrac{f(z)}{f(x)}$. Combining these, we get $f(xz)^2 = 1$ and hence $f(y)^2 = 1$ for all $y \neq 0$. Taking $x = z$, we also see that $f(x^2) = 1$ and we conclude that f takes all positive reals to 1. Since $f(x)^2 = 1$ for all $x \neq 0$, we also have $f(x) = \pm 1$ for all $x \neq 0$. Taking $x = y^2$ and $z = -1$, we see that

$f(-y^2) = 1/f(-1)$, so that $f(x)$ has the same sign as that of $f(-1)$ for all $x < 0$. Thus we get two functions:

$$f_1(x) = \begin{cases} 0 & \text{for } x = 0, \\ 1 & \text{for } x \neq 0; \end{cases}$$

$$f_2(x) = \begin{cases} 0 & \text{for } x = 0, \\ 1 & \text{for } x > 0, \\ -1 & \text{for } x < 0 \end{cases}$$

It is easy to check that these functions satisfy the requirement of our problem.

Suppose $f(zx) = f(z)f(x)$, for all $x \neq 0, z \neq 0$. This implies that $f(1/x) = 1/f(x)$ for all $x \neq 0$ and hence

$$f\left(\frac{x^2+y^2}{2xy}\right) = \frac{f(x^2+y^2)}{f(2)f(x)f(y)},$$

for all $x, y \neq 0$. Comparing this with (1), we see that,

$$f(x^2+y^2) = \frac{f(2)}{2}\Big\{f(x)^2 + f(y)^2\Big\}. \tag{2}$$

Taking $x = 2, y = 1$, we get

$$f(5) = \frac{f(2)}{2}\Big\{f(2)^2 + 1\Big\}. \tag{3}$$

Similarly $x = 3, y = 1$ leads, after using $f(10) = f(5)f(2)$, to the relation

$$f(5) = \frac{1}{2}\Big\{f(3)^2 + 1\Big\}. \tag{4}$$

Now taking $x = 3, y = 4$, we also get

$$f(5)^2 = \frac{f(2)}{2}\Big\{f(3)^2 + f(4)^2\Big\} = \frac{f(2)}{2}\Big\{f(3)^2 + f(2)^4\Big\}. \tag{5}$$

Eliminating $f(5)$ and $f(3)$ from relations (3), (4) and (5), we get

$$\big\{f(2) - 2\big\}\big\{f(2)^4 - 1\big\} = 0.$$

There are three possibilities: $f(2) = \pm 1$ and $f(2) = 2$.
(A) Suppose $f(2) = 1$. In this case (2) reduces to

$$\begin{aligned} f(x^2+y^2) &= \frac{1}{2}\Big\{f(x)^2 + f(y)^2\Big\} \\ &= \frac{1}{2}\Big\{f(x^2) + f(y^2)\Big\}, \ x, y \neq 0. \end{aligned}$$

Thus we get $f(x+y) = \big(f(x) + f(y)\big)/2$ for all $x > 0$ and $y > 0$. Replacing y by $y+z$, we get

$$f\big(x+(y+z)\big) = \frac{1}{2}\Big\{f(x)+f(y+z)\Big\} = \frac{f(x)}{2} + \frac{f(y)}{4} + \frac{f(z)}{4}.$$

But we can write $f\big(x+(y+z)\big) = f\big((x+y)+z\big)$, and the second representation gives

$$f\big((x+y)+z\big) = \frac{f(x)}{4} + \frac{f(y)}{4} + \frac{f(z)}{2}.$$

Comparing these two we see that $f(x) = f(z)$ for all $x > 0, z > 0$. Thus f is constant on positive reals and this constant is equal to $f(1) = 1$. Since $f(x)^2 = f\big(x^2\big) = 1$, we get the same solutions f_1 and f_2.

If $f(2) = -1$, the same analysis shows that f is constant on positive reals. However $f(1) = 1$ and $f(2) = -1$ are incompatible. Thus we do not get any solution in this case.
(B) Suppose $f(2) = 2$. In this case we get $f\big(x^2+y^2\big) = f(x)^2 + f(y)^2 = f\big(x^2\big) + f\big(y^2\big)$, so that

$$f(x+y) = f(x) + f(y), \tag{6}$$

for all positive reals x, y. Observe that $f\big(x^2\big) = f(x)^2 > 0$ for all $x \neq 0$. Thus for $y > x > 0$, we have

$$f(y) = f(y-x+x) = f(y-x) + f(x) > f(x).$$

This shows that f is strictly increasing on positive reals. Using (6) it is easy to prove that $f(rx) = rf(x)$ for all

positive rationals r and $x > 0$. Thus $f(r) = rf(1) = r$ for all positive rationals. This along with strict increasing nature of f implies that $f(x) = x$ for all $x > 0$. Since $f(-1)^2 = f(1) = 1$, and $f(-x) = f(-1)f(x)$, there are two possibilities: $f(-x) = f(x)$ and $f(-x) = -f(x)$. Each leads to one solution: $f_3(x) = x$ for all x and $f_4(x) = |x|$ for all x. Again we check that these are indeed solutions. ■

Problem 3.23 Find all functions $f : [1, \infty) \to [1, \infty)$ which satisfy

(a) $f(x) \leq 2(1 + x)$ for all $x \in [1, \infty)$;

(b) $xf(x + 1) = f(x)^2 - 1$ for all $x \in [1, \infty)$.

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Solution: It is easy to verify that $f(x) = x + 1$ satisfies both (a) and (b). We show that this is the only solution. We have

$$\begin{aligned} f(x)^2 = xf(x+1) + 1 &\leq x\big(2(x+2)\big) + 1 \\ &= 1 + 4x + 2x^2 \\ &< 2\big(1 + 2x + x^2\big) = 2(1+x)^2. \end{aligned}$$

It follows that $f(x) < \sqrt{2}\big(1 + x\big)$. Using this fresh bound, we obtain

$$\begin{aligned} f(x)^2 = xf(x+1) + 1 &< \sqrt{2}x\big(2 + x\big) + 1 \\ &= \sqrt{2}x^2 + 2\sqrt{2}x + 1 \\ &< \sqrt{2}\big(x^2 + 2x + 1\big) \\ &= \sqrt{2}\big(x + 1\big)^2. \end{aligned}$$

Thus we obtain another bound; $f(x) < 2^{1/4}\big(x + 1\big)$. Continuing by induction, we arrive at

$$f(x) < 2^{1/2^k}\big(1 + x\big),$$

for all $k \in \mathbb{N}$, and $x \in [1, \infty)$. It follows that $f(x) \leq 1 + x$ for all $x \in [1, \infty)$.

Suppose $f(x_0) < 1 + x_0$ for some $x_0 \in [1, \infty)$. Let $f(x_0) = 1 + x_0 - \epsilon$ where $0 < \epsilon \leq x_0$. We then have

$$\begin{aligned} f(1 + x_0) = \frac{f(x_0)^2 - 1}{x_0} &= \frac{(1 + x_0 - \epsilon)^2 - 1}{x_0} \\ &= x_0 - 2\epsilon + 2 + \frac{\epsilon^2 - 2\epsilon}{x_0} \\ &\leq x_0 - 2\epsilon + 2 + \epsilon - 2 \\ &= x_0 - \epsilon < x_0. \end{aligned}$$

Using this bound we get

$$\begin{aligned} f(x_0 + 2) = \frac{f(x_0 + 1)^2 - 1}{x_0 + 1} &< \frac{x_0^2 - 1}{x_0 + 1} \\ &< x_0 - 1. \end{aligned}$$

This in turn implies that

$$\begin{aligned} f(x_0 + 3) = \frac{f(x_0 + 2)^2 - 1}{x_0 + 2} &< \frac{(x_0 - 1)^2 - 1}{x_0 + 2} \\ &= \frac{x_0(x_0 - 2)}{x_0 + 2} < x_0 - 2. \end{aligned}$$

By an easy induction, we see that

$$f(x_0 + k) < x_0 - k + 1.$$

If k is large enough, then $f(x_0 + k) < 1$. This contradiction forces $f(x) = 1 + x$ for all $x \in [1, \infty)$. ∎

In some cases the functional relation may reveal some useful information about the function.

Problem 3.24 Let $f : [0, 1] \to \mathbb{R}$ be such that $f(x) \geq 0$ for all $x \in [0, 1]$, $f(1) = 1$ and

$$f(x) + f(y) \leq f(x + y), \tag{1}$$

for all $x, y \in [0,1]$ such that $x+y \in [0,1]$. Prove that

$$f(x) \leq 2x,$$

for all $x \in [0,1]$.

Solution: We show, in fact, that $f(x) < 2x$ except for $x = 0$. Putting $y = 1 - x$ in (1), we obtain

$$f(x) + f(1-x) \leq f(1) = 1.$$

Since $f(x) \geq 0$ for all $x \in [0,1]$, it follows that $f(x) \leq 1$ for all $x \in [0,1]$. Taking $x = 0$ in the above inequality, we see that

$$f(0) + f(1) \leq f(1),$$

and hence $f(0) \leq 0$. It follows that $f(0) = 0$.

Taking $y = x$ in (1), we obtain $2f(x) \leq f(2x)$ for all $x \in [0, 1/2]$. By induction, we obtain

$$2^n f(x) \leq f\big(2^n x\big),$$

for all x such that $2^n x \in [0,1]$; i.e., for $x \in [0, 2^{-n}]$. If $x > 1/2$, then

$$f(x) \leq 1 < 2x.$$

Suppose $0 < x \leq 1/2$. Choose $n \geq 1$ such that $\dfrac{1}{2^{n+1}} < x \leq \dfrac{1}{2^n}$. For this choice of n, we have $2^n x \in [0,1]$ and hence

$$2^n f(x) \leq f\big(2^n x\big) \leq 1 < 2^{n+1} x = 2^n\big(2x\big),$$

by our choice of n. It follows that

$$f(x) < 2x,$$

in this case as well. Thus $f(x) < 2x$ for all $x \in [0,1]$, $x \neq 0$.

■

Problem 3.25 Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function such that

(a) $f(x) \geq 0$, $\forall x \in \mathbb{R}^n$;

(b) $\{f(x+y)\}^2 + \{f(x-y)\}^2 = 2\{f(x)^2 + f(y)^2\}$,

for all $x, y \in \mathbb{R}^n$. Prove that

$$f(x+y) \leq f(x) + f(y),$$

for all $x, y \in \mathbb{R}^n$.

Solution: Putting $x = y = 0$ in (b), we get $2f(0)^2 = 4f(0)^2$ and hence $f(0) = 0$. Putting $x = 0$ in (b) and using $f(0) = 0$ we see that $f(y)^2 = f(-y)^2$. Since f is nonnegative we conclude that $f(y) = f(-y)$, for all $y \in \mathbb{R}^n$.

Define $g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by,

$$g(x,y) = \frac{1}{4}\Big(f(x+y)^2 - f(x-y)^2\Big).$$

We observe that

$$g(x,x) = \frac{1}{4}f(2x)^2 = f(x)^2$$

using (b). Moreover we also observe that

$$\begin{aligned} g(x,y) &= \frac{1}{4}\Big(f(x+y)^2 - f(x-y)^2\Big) \\ &= \frac{1}{4}\Big(f(y+x)^2 - f(y-x)^2\Big) \\ &= g(y,x), \end{aligned}$$

where we have used $f(z)^2 = f(-z)^2$. Now we consider

$$\begin{aligned} g(x+z,y) &= \frac{1}{4}\Big(f(x+z+y)^2 - f(x+z-y)^2\Big) \\ &= \frac{1}{4}\Big(2f(x+y)^2 + 2f(z)^2 \\ &\qquad - f(x+y-z)^2 - f(x+z-y)^2\Big) \\ &= \frac{1}{4}\Big(2f(x+y)^2 + 2f(z)^2 \\ &\qquad - \Big\{f(x+y-z)^2 + f\big(x-(y-z)\big)^2\Big\}\Big) \\ &= \frac{1}{4}\Big(2f(x+y)^2 + 2f(z)^2 \\ &\qquad - 2f(x)^2 - 2f(y-z)^2\Big). \end{aligned}$$

Using (b), we also obtain

$$\begin{aligned} 2f(x+y)^2 - 2f(x)^2 &= f(x+y)^2 + f(x+y)^2 - 2f(x)^2 \\ &= f(x+y)^2 + 2f(y)^2 - f(x-y)^2, \end{aligned}$$

and

$$\begin{aligned} 2f(z)^2 - 2f(y-z)^2 &= 2f(z)^2 - f(y-z)^2 - f(y-z)^2 \\ &= f(y+z)^2 - 2f(y)^2 - f(y-z)^2. \end{aligned}$$

Using these simplifications in the expression for $g(x+z,y)$, we obtain

$$\begin{aligned} g(x+z,y) &= \frac{1}{4}\Big(f(x+y)^2 + 2f(y)^2 - f(x-y)^2 \\ &\qquad + f(y+z)^2 - 2f(y)^2 - f(y-z)^2\Big) \\ &= \frac{1}{4}\Big(f(x+y)^2 - f(x-y)^2\Big) \\ &\qquad + \frac{1}{4}\Big(f(y+z)^2 - f(y-z)^2\Big) \\ &= g(x,y) + g(y,z) = g(x,y) + g(z,y). \end{aligned}$$

Thus g is symmetric and additive in the first variable (and hence also in the second variable).

Using the additivity it is easy to show that $g(nx, y) = ng(x, y)$ for each $n \in \mathbb{N}$. Since $g(0, y) = 0$, we see that $g(-x, y) = -g(x, y)$. Thus we can obtain $g(nx, y) = ng(x, y)$ for all $n \in \mathbb{Z}$. Using this, it is not hard to get $g(rx, y) = rg(x, y)$ for all $r \in \mathbb{Q}$.

Take any $r \in \mathbb{R}$. We have

$$\begin{aligned} 0 \leq f(rx + y)^2 &= g(rx + y, rx + y) \\ &= r^2 g(x, x) + 2rg(x, y) + g(y, y). \end{aligned}$$

Since this is true for every rational number, the discriminant must be non-positive. This condition gives

$$g(x, y)^2 \leq g(x, x)g(y, y).$$

This is same as $|g(x, y)| \leq f(x)f(y)$. (Here we use the fact that f is nonnegative.) We thus get

$$\begin{aligned} f(x + y)^2 &= g(x + y, x + y) \\ &= g(x, x) + 2g(x, y) + g(y, y) \\ &\leq f(x)^2 + 2f(x)f(y) + f(y)^2 \\ &= \big(f(x) + f(y)\big)^2. \end{aligned}$$

Again using the non-negativity of f we obtain,

$$f(x + y) \leq f(x) + f(y).$$

∎

Problem 3.26 Find all functions $f : [0, 1] \times [0, 1] \to [0, 1]$ satisfying the following conditions:

(a) $f(f(x, y), z) = f(x, f(y, z))$, for all $x, y, z \in [0, 1]$;

(b) $f(x, y) = f(y, x)$;

(c) $f(x,1) = x$, for all $x \in [0,1]$;

(d) $f(zx, zy) = z^k f(x,y)$ for all $x, y, z \in [0,1]$ where k is a fixed positive real number.

(AMM-1988)

Solution: We observe that $f(x,y) = \min(x,y)$ and $f(x,y) = xy$ both satisfy all the four conditions of the problem. We show that these are the only functions which are solutions for the given problem.

Taking $x = 0$ in (c), we get $f(0,1) = 0$. This implies from (b) and (d) that

$$f(x,0) = f(0,x) = x^k f(0,1) = 0.$$

Thus $f(x,0) = 0$ and $f(x,1) = 1$ for all $x \in [0,1]$. Using the symmetry condition (b), we see that f is now completely determined on the boundary of the square $[0,1] \times [0,1]$. Suppose $0 < x \leq y < 1$. Then we have

$$\begin{aligned} f(x,y) = f(y,x) &= y^k f\left(1, \frac{x}{y}\right) \\ &= y^k f\left(\frac{x}{y}, 1\right) = y^k \left(\frac{x}{y}\right) = y^{k-1}x. \end{aligned}$$

This shows that

$$f(u,v) = \min(u,v)(\max(u,v))^{k-1},$$

for all points (u,v) in the interior of the square $[0,1] \times [0,1]$. It is easy to check that the above relation is valid on the boundary of $[0,1] \times [0,1]$ as well.

We show that $k = 1$ or 2. Choose any $y \in [1/2, 1]$ and x such that $0 \leq x \leq \min\{1, y^{k-1}, 2^k y\}/2$. Using the condition (a), we see that $f(f(x,1/2),y) = f(x, f(1/2,y))$.

But $x \leq 1/2 \leq y$ and hence $f(x, 1/2) = x(1/2)^{k-1}$ and $f(1/2, y) = y^{k-1}/2$. Using these values, we obtain

$$f\left(x\left(\frac{1}{2}\right)^{k-1}, y\right) = f\left(x, \frac{1}{2}y^{k-1}\right).$$

But the choice of x shows that $x \leq 2^{k-1}y$ and hence

$$f\left(x\,(1/2)^{k-1}, y\right) = x(1/2)^{k-1}y^{k-1}.$$

Similarly, the choice of x gives $f\left(x, \frac{1}{2}y^{k-1}\right) = x(y^{k-1}/2)^{k-1}$. Comparing these two we obtain $k-1 = (k-1)^2$. We conclude that $k = 1$ or 2.

These choices of k lead to two different solutions. If $k = 1$, then we get $f(x, y) = \min(x, y)$; if $k = 2$, then we get $f(x, y) = \min(x, y)\max(x, y) = xy$. ∎

Problem 3.27 Find all functions $f : \mathbb{R}_0 \to \mathbb{R}_0$ which satisfy the equation

$$f(x^2 + y^2) = f(x)^2 + f(y)^2, \tag{1}$$

for all $x, y \in \mathbb{R}_0$.

Solution: Let us take $x = y = 0$ in (1). We obtain $f(0) = 2f(0)^2$ showing that $f(0) = 0$ or $f(0) = 1/2$. We consider these cases separately.

Case 1. Suppose $f(0) = 0$. Taking $y = 0$ in (1), we obtain

$$f(x^2) = f(x)^2, \tag{2}$$

and hence $f(x^2 + y^2) = f(x)^2 + f(y)^2$. We conclude that

$$f(x + y) = f(x) + f(y), \tag{3}$$

for all $x, y \in \mathbb{R}_0$. The relation (3) implies that $f(x) \leq f(y)$ whenever $x \leq y$. We also see from (2) that $f(1) = f(1)^2$

giving $f(1) = 0$ or 1. If $f(1) = 0$, then we can prove using (3) and induction that $f(n) = 0$ for all nonnegative integers n. Given any $x \in \mathbb{R}_0$, we can find a nonnegative integer n such that $n \leq x \leq n+1$. The monotonicity of f shows that $0 = f(n) \leq f(x) \leq f(n+1) = 0$. It follows that $f(x) = 0$ for all $x \in \mathbb{R}_0$.

If $f(1) = 1$, then again we prove by induction and using (3) that $f(n) = n$ for all nonnegative integers. By a standard argument, we can show that $f(r) = r$ for all nonnegative rationals r. If $f(x) < x$ for some x, choose a rational r such that $f(x) < r < x$. Using the monotonicity of f, we obtain $r = f(r) \leq f(x)$. This contradicts the choice of r. Similarly, we can rule out $f(x) > x$. We conclude that $f(x) = x$ for all $x \in \mathbb{R}_0$.

Case 2. Consider the possibility $f(0) = 1/2$. Taking $y = 0$ in (1), we obtain

$$f(x^2) = f(x)^2 + \frac{1}{4}. \tag{4}$$

Thus we get from (1), the relation

$$f(x^2 + y^2) = f(x^2) + f(y^2) - \frac{1}{2}.$$

This shows that

$$f(x+y) = f(x) + f(y) - \frac{1}{2}, \tag{5}$$

for all $x, y \in \mathbb{R}_0$. Taking $x = 1$ in (4), we obtain $f(1) = f(1)^2 + 1/4$ giving $(f(1) - (1/2))^2 = 0$. Thus $f(1) = 1/2$. Using (5), we also get $f(2x) = 2f(x) - \frac{1}{2}$. This leads to

$$f(3x) = f(x + 2x) = f(x) + f(2x) - \frac{1}{2} = 3f(x) - 1.$$

By an easy induction, we obtain

$$nf(x) - \frac{(n-1)}{2},$$

for all natural numbers n. Since $f(x) \geq 0$ for all $x \in \mathbb{R}_0$, we conclude that

$$f(x) \geq \frac{n-1}{2n},$$

for all $x \in \mathbb{R}_0$. But this is true for every $n \in \mathbb{N}$. It follows that $f(x) \geq 1/2$ for all $x \in \mathbb{R}_0$. Using (5), we see that

$$f(x+y) \geq f(x),$$

for all $x, y \in \mathbb{R}_0$. This shows that f is a nondecreasing function on $\mathbb{R}_0$. Taking $x = 1$ in (6), we see that $f(n) = 1/2$ for all natural numbers. Given any $x \in \mathbb{R}_0$, choose $n \geq 0$ such that $n \leq x \leq n+1$. Using the nondecreasing nature of f and the fact that f takes the value $1/2$ at all nonnegative integers, we conclude that $f(x) = 1/2$.

Alternately, the second case can also be handled as follows. Replacing x by $x+y$ in (4), where $x, y \in \mathbb{R}_0$, we see that

$$f\left(x^2+y^2+2xy\right) = \left(f(x+y)\right)^2 + \frac{1}{4}.$$

Using (5) here, we obtain

$$f(2xy) = 2f(x)f(y) - f(x) - f(y) + 1,$$

for all $x, y \in \mathbb{R}_0$. Taking $y = 1/2$ in this relation, we get

$$2f(x)\left(1 - f\left(\frac{1}{2}\right)\right) = \left(1 - f\left(\frac{1}{2}\right)\right).$$

Taking $x = y = 1/2$ in (1), we see that

$$f\left(\frac{1}{2}\right) = 2\left(f\left(\frac{1}{2}\right)\right)^2,$$

showing that $f(1/2) \neq 1$. Thus we conclude that $2f(x) = 1$ for all $x \in \mathbb{R}_0$.

Thus there are three solutions to the given equation: $f(x) \equiv 0$; $f(x) \equiv 1/2$; and $f(x) = x$.

■

Problem 3.28 Find all functions $f : \mathbb{R} \to \mathbb{R}$ which satisfy the equation

$$f\big(x^2 + y + f(y)\big) = 2y + f(x)^2, \tag{1}$$

for all real numbers x, y.

(AMM-2001)

Solution: We observe that f is on to. Hence we can choose β such that $f(\beta) = 0$. Let us put $f(0) = s$. Taking $y = \beta$ in (1), we obtain

$$f\big(x^2 + \beta\big) = 2\beta + f(x)^2. \tag{2}$$

Changing x to $-x$ in (2), we see that $f(x)^2 = f(-x)^2$ for all x. Hence $f(-x) = f(x)$ or $f(-x) = -f(x)$ for each x. In both the cases $f(-\beta) = 0$. Now (1) also gives

$$f\big(y + f(y)\big) = 2y + s^2, \tag{3}$$

for all y. In particular $0 = f(\beta) = 2\beta + s^2$ and $0 = f(-\beta) = -2\beta + s^2$. It follows that $\beta = 0$ and $s = 0$. Thus we infer that $f(0) = 0$ and $f(x) = 0$ implies that $x = 0$. Now (1) gives

$$f\big(x^2\big) = f(x)^2, \quad \forall x \in \mathbb{R}; \tag{5}$$

$$f\big(y + f(y)\big) = 2y, \quad \forall y \in \mathbb{R}. \tag{6}$$

Now (5) shows that $f(x) \geq 0$ for all $x \geq 0$. Take any $z < 0$ and write $z = -x^2$. Put $f(x)^2 = f\big(x^2\big) = 2y$. Then we see that $-2y + f(x)^2 = 0$ and hence

$$0 = -2y + f(x)^2 = f\big(x^2 - y + f(-y)\big).$$

However, we have seen earlier that $f(y) = 0$ implies that $y = 0$. Therefore we have $x^2 - y + f(-y) = 0$. This leads to $-y + f(-y) = -x^2$ and hence

$$f\big(-x^2\big) = f\big(-y + f(-y)\big) = -2y = -f(x)^2 = -f\big(x^2\big).$$

We conclude that $f(-z) = -f(z)$ and hence f is an *odd* function. Changing y to $y + f(y)$ in (6), we get

$$f\big(3y + f(y)\big) = 2\big(y + f(y)\big).$$

If $y \geq 0$, then we also have

$$\begin{aligned} f\big(3y + f(y)\big) &= f\big(2y + y + f(y)\big) \\ &= 2y + f\big(\sqrt{2y}\,\big)^2 = 2y + f(2y). \end{aligned}$$

Thus we obtain $f(2y) = 2f(y)$ for all $y \geq 0$. Since f is an *odd* function, this is also true for all $y < 0$. We hence have

$$f(2y) = 2f(y), \quad \text{for all } \ y \in \mathbb{R}. \tag{7}$$

Since f takes nonnegative reals to nonnegative reals and f is an *odd* function, f maps negative reals to negative reals. In particular $yf(y) \geq 0$ for all real y. Putting $x = y + f(y)$ in (5), we get

$$f\Big(\big(y + f(y)\big)^2\Big) = \Big[f\big(y + f(y)\big)\Big]^2 = 4y^2.$$

On the other hand, we see that

$$\begin{aligned} f\Big(\big(y + f(y)\big)^2\Big) &= f\Big(y^2 + 2yf(y) + f(y)^2\Big) \\ &= f\Big(2yf(y) + y^2 + f\big(y^2\big)\Big) \\ &= 2y^2 + f\big(\sqrt{2yf(y)}\,\big)^2 \\ &= 2y^2 + f\big(2yf(y)\big). \end{aligned}$$

Comparing the two expressions and using (7). we get $f\big(yf(y)\big) = y^2$. Taking $y = x + f(x)$ in this relation, we obtain

$$\begin{aligned} \big(x + f(x)\big)^2 &= f\Big(\big(x + f(x)\big)f\big(x + f(x)\big)\Big) \\ &= f\Big(2x\big(x + f(x)\big)\Big) \\ &= 2f\big(x^2 + xf(x)\big) \\ &= 2f\Big(xf(x) + f\big(xf(x)\big)\Big) = 4xf(x). \end{aligned}$$

This reduces to $\big(x - f(x)\big)^2 = 0$ and hence $f(x) = x$ for all real x.

■

Exercises

3.1 In the equations below f is a function from $\mathbb{R}$ to $\mathbb{R}$. Find f:

(i) $f(x+y) - 2f(x-y) + f(x) - 2f(y) = y - 2$;

(ii) $f(x+y) + 2f(x-y) + f(x) + 2f(y) = 4x + y$;

(iii) $f(x)f(x+y) = f(y)^2 f(x-y)^2 e^{y+4}$;

(iv) $f(x+y) + f(x-y) - (y+2)f(x) + y(x^2 - 2y) = 0$.

3.2 Does there exist a function $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(1 + f(x)) = 1 - x, \quad \text{and} \quad f(f(x)) = x?$$

3.3 Suppose $f : \mathbb{R} \to \mathbb{R}$ is such that

$$f(xy) = xf(x) + yf(y),$$

for all $x, y \in \mathbb{R}$. Prove that $f(x) = 0$ for all $x \in \mathbb{R}$.

3.4 Find all functions $f : \mathbb{R} \setminus \{0, 1\} \to \mathbb{R}$ such that

$$f(x) + f\left(\frac{1}{1-x}\right) = x,$$

for all $x \in \mathbb{R} \setminus \{0, 1\}$.

3.5 Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(f(x) + yz) = x + f(y)f(z),$$

for all reals x, y, z.

3.6 Find all $f : \mathbb{R} \to \mathbb{R}$ which satisfy the equation

$$f(y + zf(x)) = f(y) + xf(z),$$

for all $x, y, z \in \mathbb{R}$.

3.7 Suppose $f : \mathbb{R} \to \mathbb{R}$ is such that

$$f(xf(z) + f(y)) = zf(x) + y,$$

for all real numbers x, y, z. Prove that $f(x) = x$ for all reals x.

3.8 Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

(a) $f(xy) = f(x)f(y)$, for all $x, y \in \mathbb{R}$;

(b) for some $z \neq 0$, $f(x + z) = f(x) + f(z)$ for all $x \in \mathbb{R}$.

3.9 Find all $f : \mathbb{R}_0 \to \mathbb{R}_0$ satisfying the conditions:

(a) $f(xf(y))f(y) = f(x + y)$, for all $x, y \geq 0$;

(b) $f(2) = 0$;

(c) $f(x) \neq 0$, for $0 \leq x < 2$.

3.10 Find all functions $f : \mathbb{C} \to \mathbb{C}$ such that

$$f(z) + zf(1 - z) = 1 + z,$$

for all $z \in \mathbb{C}$.

3.11 Find all $f : \mathbb{Q}^+ \to \mathbb{Q}^+$ satisfying

(a) $f(x) + f\left(\frac{1}{x}\right) = 1$, for all $x \in \mathbb{Q}^+$;

(b) $f(f(x)) = \frac{f(x+1)}{f(x)}$, for all $x \in \mathbb{Q}^+$.

3.12 Find all $f : \mathbb{R} \to \mathbb{R}$ which obey the relation

$$(x-y)f(x+y) - (x+y)f(x-y) = 4xy(x^2 - y^2),$$

for all real numbers x, y.

3.13 Find all functions $f : (0, \infty) \to (0, \infty)$ such that

(a) $f(x) \in (1, \infty)$ for each $x \in (0, 1)$;

(b) $f(xf(y)) = yf(x)$, for all $x, y \in (0, \infty)$.

3.14 Prove that there is a unique function $f : \mathbb{R}_0 \to \mathbb{R}_0$ such that

$$f(f(x)) = 12x - f(x), \quad \text{for all } x \in \mathbb{R}_0.$$

3.15 Find all functions $f : \mathbb{R} \to [0, \infty)$ such that

$$f(x^2 + y^2) = f(x^2 - y^2) + f(2xy),$$

for all $x, y \in \mathbb{R}$.

3.16 Find all pairs of functions $f, g : \mathbb{R} \to \mathbb{R}$ which satisfy

(a) g is an one-one function;

(b) $f(g(x) + y) = g(x + f(y))$, for all $x, y \in \mathbb{R}$.

3.17 Determine all functions $f : \mathbb{R} \to \mathbb{R}$ which satisfy the equation

$$xf(x) - yf(y) = (x - y)f(x + y), \quad \text{for all } x, y \in \mathbb{R}.$$

3.18 Let α be a given real number. Find all functions $f : (0, \infty) \to (0, \infty)$ satisfying the relation

$$\alpha x^2 f\left(\frac{1}{x}\right) + f(x) = \frac{x}{1 + x},$$

for all $x \in (0, \infty)$.

(Israel Mathematical Olympiads-1995)

3.19 Let $g : \mathbb{C} \to \mathbb{C}$ be a given function, $a \in \mathbb{C}$ and ω be primitive cube root of unity. Find all $f : \mathbb{C} \to \mathbb{C}$ which satisfy

$$f(z) + f(\omega z + a) = g(z),$$

for all $z \in \mathbb{C}$.

3.20 Find all $f : \mathbb{R} \to \mathbb{R}$ which satisfy

$$f(x + y) = f(x)f(y)f(xy),$$

for all $x, y \in \mathbb{R}$.

3.21 Find all $f : \mathbb{R} \to \mathbb{R}$ such that

$$f\big(f(x)^2 + y\big) = x^2 + f(y), \quad \text{for all } x, y \in \mathbb{R}.$$

3.22 Let $n \geq 2$ be a natural number. Find all $f : \mathbb{R} \to \mathbb{R}$ such that

$$f\big(x + y^n\big) = x + f(y)^n, \quad \text{for all } x, y \in \mathbb{R}.$$

3.23 Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x+y) - f(x-y) = f(x)f(y), \quad \text{for all } x, y \in \mathbb{R}.$$

3.24 Find all functions $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ which satisfy the following conditions:

(a) $f(x+u, y+u) = f(x,y) + u$, for all $x, y, u \in \mathbb{R}$;

(b) $f(xu, yu) = f(x,y)u$, for all $x, y, u \in \mathbb{R}$.

3.25 Find all $f, g : \mathbb{R} \to \mathbb{R}$ such that

$$f\big(x + g(y)\big) = xf(y) - yf(x) + g(x),$$

for all reals x, y. (IMO-2000 Short-List)

3.26 Find all $f : \mathbb{R} \to \mathbb{R}$ such that

$$f\big(f(x+y)\big) = f(x+y) + f(x)f(y) - xy,$$

for all reals x, y. (Belarusian Mathematical Olympiad-1995)

3.27 Let $\mathbb{Q}^+$ denote the set of all positive rational numbers. Find all functions $f : \mathbb{Q}^+ \to \mathbb{Q}^+$ such that

(a) $f(x+1) = f(x) + 1$;

(b) $f\big(x^2\big) = \big(f(x)\big)^2$.

(Ukrainian Olympiad-1997)

3.28 A function $f : \mathbb{N} \to \mathbb{R}$ satisfies for some positive integer m the conditions

$$f(m) = f(1995), f(m+1) = f(1996), f(m+2) = f(1997),$$

$$f(n+m) = \frac{f(n) - 1}{f(n) + 1}, \quad \text{for all positive integers } n.$$

Prove that $f(n+4m) = f(n)$ for all $n \in \mathbb{N}$ and find the least value of m for this is true.

(Nordic Mathematical Contest-1999)

3.29 Suppose a function $f : \mathbb{R} \to \mathbb{R}$ satisfies the conditions

$$f(x+19) \leq f(x)+19 \text{ and } f(x+94) \geq f(x)+94,$$

for all real numbers x. Prove that $f(x+1) = f(x)+1$ for all $x \in \mathbb{R}$.

(Austrian-Polish Mathematics Competition-1994)

3.30 Find all functions $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ satisfying the equation

$$\frac{1}{x} f(-x) + f\left(\frac{1}{x}\right) = x,$$

for all $x \neq 0$.

3.31 Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$\frac{1}{2} f(xy) + \frac{1}{2} f(xz) - f(x) f(yz) \geq \frac{1}{4},$$

holds for all reals x, y, z.

(Vietnamese National Olympiad-1991)

3.32 Find all functions $f : \mathbb{R} \to \mathbb{R}$ which are such that

$$f(x+y) + f(xy) = f(x) f(y) + 1,$$

for all $x, y \in \mathbb{R}$. (See problem 17 of Chapter 3.)

3.33 Suppose $f : \mathbb{Q} \to \{0, 1\}$ is such that $f(0) = 0$, $f(1) = 1$ and

$$f(x) = f(y) \implies f(x) = f(y) = f\left(\frac{x+y}{2}\right),$$

for all $x, y \in \mathbb{Q}$. Prove that $f(x) = 1$ for all rationals $x \geq 1$.

3.34 Find all functions $f, g : \mathbb{R} \to \mathbb{R}$ such that

$$\frac{xf(y) - yf(x)}{x - y} = g(x + y),$$

for all $x \neq y$.

3.35 Find all functions $f, g, h : \mathbb{R} \to \mathbb{R}$ such that

$$\frac{xf(y) - yg(x)}{x - y} = h(x + y),$$

for all $x \neq y$.

4

Cauchy's Equation and Other Problems

One of the classical equations posed on the real line $\mathbb{R}$ is what is generally known as Cauchy's equation. The problem is to characterize all functions $f : \mathbb{R} \to \mathbb{R}$ which obey the functional equation

$$f(x+y) = f(x) + f(y), \tag{1}$$

for all real numbers x, y. Such a function f is called an *additive* function. Thus we have to completely characterize the class of all additive functions on $\mathbb{R}$. This problem looks simple but lies in the heart of the structures of $\mathbb{Q}$ and $\mathbb{R}$. Let us start analysing it till we cannot make any further progress.

By putting $x = y = 0$ in (1), we see that $f(0) = 0$. If we take $y = -x$ in (1) and invoke $f(0) = 0$, we obtain

$$f(x) + f(-x) = f(0) = 0,$$

so that $f(-x) = -f(x)$ for all $x \in \mathbb{R}$. Thus $f(x)$ is an *odd* function and it is sufficient to determine $f(x)$ for positive real numbers x. By taking $y = x$ in (1), we obtain $f(2x) = 2f(x)$. If we put $y = 2x$ in (1), we see that $f(3x) = 3f(x)$. An easy induction proves that

$$f(nx) = nf(x), \tag{2}$$

for all natural numbers n and real numbers x. Taking $x = 1$ in (2) and writing $c = f(1)$, we obtain $f(n) = cn$ for all natural numbers. Since f is an *odd* function, we also obtain $f(n) = cn$ for all integers n.

Let us take a rational number $r = p/q$, where p is an integer and q is a natural number. Then

$$cp = f(p) = f(qr) = qf(r);$$

we have used the fact that $f(n) = cn$ for all integers and (2). This shows that

$$f(r) = c\left(\frac{p}{q}\right) = cr. \tag{3}$$

Thus we have determined the value of f at all rational points: $f(r) = cr$ for all $r \in \mathbb{Q}$.

Now we come to the first serious encounter with $\mathbb{R}$. The structure of $\mathbb{Q}$, though complicated compared to $\mathbb{N}$ or $\mathbb{Z}$, is not really something which cannot be handled. This is because each rational number is a quotient of two integers, the denominator non-vanishing. This additional information has helped us in determining f at rational points. Unfortunately, the structure of real numbers is not that simple when contrasted with the rational numbers. There is a *leap* from $\mathbb{Q}$ to $\mathbb{R}$. If you conjecture that $f(x) = cx$ for all real x by the information gathered on $\mathbb{Q}$, you are wrong. Using some advanced ideas, we can show that the conjecture fails without any further restrictions on the function f.

Consider a Hamel basis H of $\mathbb{R}$ over $\mathbb{Q}$; such a basis exists. Every element of $\mathbb{R}$ can be uniquely expressed as a finite linear combination of of elements of H with rational coefficients: i.e., for each x in $\mathbb{R}$, we can find a unique set $\{h_1, h_2, \cdots, h_k\}$ of elements of H and a unique set $\{r_1, r_2, \ldots, r_k\}$ of rational numbers such that

$$x = r_1h_1 + r_2h_2 + \cdots + r_kh_k.$$

We define f arbitrarily on elements of H and extend it to $\mathbb{R}$ by setting

$$f(x) = r_1f(h_1) + r_2f(h_2) + \cdots + r_kf(h_k).$$

Then it is easy to verify that f is additive on $\mathbb{R}$. Take any two basis elements h_1 and h_2 and set $f(h_1) = h_2$ and $f(h_2) = h_1$, and put $f(h) = 1$ for all other

h in H. We claim that such a function f is not of the form $f(x) = cx$ for all $x \in \mathbb{R}$. Suppose $f(x) = cx$ for some constant c and for all real numbers x. Then we get.

$$ch_1 = f(h_1) = h_2, \quad ch_2 = f(h_2) = h_1$$

It follows that $c = \pm 1$. But then $h_1 = h_2$ or $h_1 = -h_2$ which is impossible since h_1, h_2 are basis elements. This also shows that there are infinitely many functions on $\mathbb{R}$ which are just additive without being linear.

Thus we see that our functional equation cannot sustain such a leap. We know that any real number can be approximated by a rational number to any preassigned degree of closeness; we say $\mathbb{Q}$ is every where dense in $\mathbb{R}$. Even such proximity of a real number to some rational number has not helped us in determining f at a real number. This calls for imposing some extra a priori condition(s) on f. We have already encountered some such conditions in chapter 3(problems 3.4 and 3.5). Those conditions helped us in efficiently exploiting existing structure on $\mathbb{R}$ to get solution(s). We shall see in this chapter how other conditions help us in determining additive functions on $\mathbb{R}$.

Problem 4.1 Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function such that

(a) $f(x+y) = f(x) + f(y)$, for all $x, y \in \mathbb{R}$;

(b) f is monotonic on $\mathbb{R}$ (i.e., f is either monotonically increasing or monotonically decreasing).

Prove that f is *linear*: i.e., there exists a constant c such that $f(x) = cx$ for all $x \in \mathbb{R}$.

Solution: The condition (a) implies that $f(r) = cr$ for all rationals r where $c = f(1)$. Suppose f is monotonically increasing on $\mathbb{R}$. For $x \leq y$, we have $f(x) \leq f(y)$. Take any real number x. We can find an increasing sequence $\langle p_n \rangle$ and a decreasing sequence $\langle q_n \rangle$ of rationals such that $p_n \to x$

and $q_n \to x$ as $n \to \infty$. Note that $p_n \leq x$ and $x \leq q_n$. Since f is monotonically increasing, we have

$$f(p_n) \leq f(x) \leq f(q_n).$$

Using $f(p_n) = cp_n$ and $f(q_n) = cq_n$, we obtain

$$cp_n \leq f(x) \leq cq_n.$$

Letting n tend to ∞, we obtain $f(x) = cx$. A similar proof works when f is monotonically decreasing. ∎

We can replace monotonicity by continuity.

Problem 4.2 Let $f : \mathbb{R} \to \mathbb{R}$ be such that

(a) $f(x + y) = f(x) + f(y)$, for all $x, y \in \mathbb{R}$;

(b) f is continuous on $\mathbb{R}$.

Prove that f is linear.

Solution: We know that the condition (a) alone implies $f(r) = cr$ for all rationals r, where $c = f(1)$ is some constant. The continuity of f says that whenever a sequence $\langle y_n \rangle$ of real numbers converges to a real number y, the sequence $\langle f(y_n) \rangle$ converges to $f(y)$.

Take any real number x. The density of $\mathbb{Q}$ in $\mathbb{R}$ helps us to find a sequence $\langle r_n \rangle$ of rational numbers converging to x. (For example, we express x in decimal notation and we can take r_n to be the truncated expression of x up to n decimal places.) Invoking the continuity of f, we see that the sequence $\langle f(r_n) \rangle$ converges to $f(x)$. But we know that $f(r_n) = cr_n$. Thus we obtain

$$f(x) = \lim_{n \to \infty} f(r_n) = \lim_{n \to \infty} cr_n = cx.$$

We conclude that f is linear. ∎

We have imposed the continuity condition on f throughout $\mathbb{R}$. But this is not needed. The additivity of f itself helps us to prove the continuity everywhere if we know that it is continuous at a single point. Suppose f is known to be continuous at some point, say x_0. Take any arbitrary point x and select a sequence $\langle x_n \rangle$ converging to x. Then the sequence $\langle x_0 + x - x_n \rangle$ converges to x_0. By continuity of f at x_0, we see that

$$\lim_{n\to\infty} f(x_0 + x - x_n) = f(x_0).$$

But the additivity of f implies that $f(x_0+x-x_n) = f(x_0)+f(x)-f(x_n)$. We thus obtain

$$\lim_{n\to\infty} \Big(f(x_0) + f(x) - f(x_n) \Big) = f(x_0).$$

It follows that

$$\lim_{n\to\infty} \Big(f(x) - f(x_n) \Big) = 0,$$

showing that f is continuous at x.

Are there milder conditions on f which would force continuity in view of additivity? There are really beautiful conditions which can be imposed on f to achieve our goal.

Problem 4.3 Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function such that

(a) $f(x+y) = f(x) + f(y)$, for all $x, y \in \mathbb{R}$;

(b) f is bounded in a neighborhood of 0: i.e., there exists a constant M and a positive real number a such that

$$|f(x)| \leq M, \quad \text{for all } x \in (-a, a)..$$

Show that f is linear.

Solution: We show that the boundedness of f in a neighborhood of 0 and additivity imply the continuity of f at 0.

Consider a sequence $\langle x_n \rangle$ of real numbers converging to 0. Given a real number $\epsilon > 0$, choose a natural number N such that $\frac{M}{N} < \epsilon$. Since $x_n \to 0$ as $n \to \infty$, we can find a natural number K such that

$$|x_n| < \frac{a}{N}, \quad \text{for all } n \geq K.$$

Thus we see that $|Nx_n| < a$, whenever $n \geq K$. It follows that Nx_n lies in $(-a, a)$ for all $n \geq K$. By the boundedness of f on $(-a, a)$, we have $|f(Nx_n)| \leq M$ for $n \geq K$. But we have seen that the additivity of f gives $f(Nx_n) = Nf(x_n)$, since N is a natural number. Combining these two, we conclude that

$$|f(x_n)| \leq \frac{M}{N}, \quad \text{for all } n \geq K.$$

Now the choice of N shows that $|f(x_n)| < \epsilon$ for all $n \geq K$. Since $\epsilon > 0$ can be taken arbitrarily small, it follows that

$$\lim_{n \to \infty} f(x_n) = 0.$$

But by the additivity of f, we also know that $f(0) = 0$. Thus we finally obtain

$$\lim_{n \to \infty} f(x_n) = f(0),$$

proving the continuity of f at 0.

Our earlier observation shows that the continuity of f at 0 with additivity force that f is continuous at every point of $\mathbb{R}$. Now the conclusion of problem 4.1 proves that f is linear. ∎

We remark here that the boundedness of f may be assumed on some interval, not necessarily in a neighborhood of

0. The additivity then implies the boundedness in a neighborhood of 0.

As an interesting application of determination of all real bounded additive function, we consider the following problem.

Problem 4.4 Find all functions $f : \mathbb{R} \to \mathbb{R}$ which satisfy

(a) $f(x+y) = f(x) + f(y)$, for all reals x, y;

(b) $f\big(p(x)\big) = p\big(f(x)\big)$ for some polynomial $p(x)$ of degree ≥ 2.

Solution: We show that f is either bounded below or bounded above on the set $\mathbb{R}_0$ of all non-negative reals. This implies that f is bounded on an interval and the above remark shows that f is linear.

Let $t \in \mathbb{Q}$, and $x, y \in \mathbb{R}$. We have

$$p\big(x+ty\big) = p\big(x\big) + \sum_{j=1}^{n} \frac{t^j}{j!} y^j p^{(j)}(x),$$

where $p^{(j)}$ denotes the j-th derivative of $p(x)$. Using the additivity of f, we obtain

$$f\big(p(x+ty)\big) = f\big(p(x)\big) + \sum_{j=1}^{n} \frac{t^j}{j!} f\Big(y^j p^{(j)}(x)\Big).$$

We have used the fact that additivity of f gives $f(qx) = qf(x)$ for all rational numbers q. But we also have

$$\begin{aligned} p\big(f(x+ty)\big) &= p\big(f(x) + tf(y)\big) \\ &= p\big(f(x)\big) + \sum_{j=1}^{n} \frac{t^j}{j!} \big(f(y)\big)^j p^{(j)}\big(f(x)\big). \end{aligned}$$

Since degree of $p(x) \geq 2$, using (b) we get the relations

$$\begin{aligned} f(yp'(x)) &= f(y)p'(f(x)); \\ f(y^2p''(x)) &= f(y)^2p''(f(x)). \end{aligned}$$

Choose a such that $p''(a) = b \neq 0$. Then we get

$$f(by^2) = cf(y)^2,$$

for some constant c. Now the additivity of f also implies that $f(-y) = -f(y)$. Hence we obtain

$$f(-by^2) = -f(by^2) = -cf(y)^2.$$

If $b > 0$, then $f(by^2) = cf(y)^2$ is either ≤ 0 or ≥ 0 according as $c \leq 0$ or $c > 0$. If $b < 0$, consider $-b$. In any case f is either bounded below or bounded above on $\mathbb{R}_0$. ∎

We now introduce some simple notions which help us to determine the class of linear additive function f on $\mathbb{R}$. An open disc $D_r(a, b)$ in $\mathbb{R}^2$ with centre (a, b) and radius r is defined as the set

$$D_r(a, b) = \left\{(x, y) \in \mathbb{R}^2 \mid (x - a)^2 + (y - b)^2 < r^2\right\}.$$

A set $E \subset \mathbb{R}^2$ is said to be *dense* in $\mathbb{R}^2$ if every every open disc in $\mathbb{R}^2$ contains a point of E. For example the set $E = \{(x, y) \in \mathbb{R}^2 \mid x, y \in \mathbb{Q}\}$ is a dense subset of $\mathbb{R}^2$.

With every function $f : \mathbb{R} \to \mathbb{R}$, we associate its *graph* $G(f)$ by

$$G(f) = \left\{(x, f(x)) \in \mathbb{R}^2 \mid x \in \mathbb{R}\right\}.$$

For example if $f(x) = ax + b$, then $G(f)$ is a straight line in $\mathbb{R}^2$. If f is an additive function which is not linear, its graph could be extremely bizarre.

Problem 4.5 Suppose $f : \mathbb{R} \to \mathbb{R}$ is an additive function which is not linear. (i.e., $f(x+y) = f(x) + f(y)$, but there is no c such that $f(x) = cx$ for all real numbers x.) Show that $G(f)$, the graph of f, is dense in $\mathbb{R}^2$.

Solution: Suppose f is additive but not linear. Let us put $c = f(1)$ and choose a real number α such that $f(\alpha) \neq c\alpha$. Define a new function g by

$$g(x) = \frac{f(x) - cx}{f(\alpha) - c\alpha}.$$

The additivity of f implies that g is also additive on $\mathbb{R}$. Moreover $g(1) = 0$. Using the additivity of g, we conclude that $g(q) = qg(1)$ for all rationals q (see problem 4.1). Thus we see that $g(q) = 0$ for all rationals q.

Consider any disc $D_r(x, y)$. Choose a rational number q such that $|q - y| < \frac{r}{2}$. This is possible by the density of $\mathbb{Q}$ in $\mathbb{R}$. Now choose a rational number p such that $|p - (x - q\alpha)| < \frac{r}{2}$. We then have

$$(p + q\alpha - x)^2 + (q - y)^2 < \frac{r^2}{4} + \frac{r^2}{4} = \frac{r^2}{2} < r^2.$$

Thus the point $(p + q\alpha, q)$ is in the disc $D_r(x, y)$. However, we observe by the additivity of g that

$$g(p + q\alpha) = g(p) + qg(\alpha) = qg(\alpha),$$

since $g(p) = 0$ for rational p. But $g(\alpha) = 1$ by the definition of g. Thus we obtain $g(p + q\alpha) = q$ showing that the point $(p + q\alpha, q)$ lies on $G(g)$, the graph of g.

This shows that every open disc in $\mathbb{R}^2$ contains a point of $G(g)$ and hence that $G(g)$ is dense in $\mathbb{R}^2$. We have to go back to f using this information. Observe that

$$f(x) = ug(x) + cx,$$

where $u = f(\alpha) - c\alpha$. Take any disc $D_r(a, b)$ in $\mathbb{R}^2$. Consider the disc D given by

$$D = D_s\left(a, \frac{1}{u}\big(b - ca\big)\right),$$

where s is given by

$$s = \sqrt{\frac{r^2}{2\beta}}, \quad \beta = \max\{2u^2, 1 + 2c^2\}.$$

Since $G(g)$ is dense in $\mathbb{R}^2$, we can find a real y such that $\big(y, g(y)\big)$ lies in D (i.e., D contains a point of $G(g)$). Now consider the point $\big(y, ug(y) + cy\big)$. We observe that

$$\begin{aligned}
&\big(a - y\big)^2 + \big(b - ug(y) - cy\big)^2 \\
&\quad = (a - y)^2 + \left\{u\left[\frac{1}{u}(b - ca) - g(y)\right] + c(a - y)\right\}^2 \\
&\quad \le (1 + 2c^2)(a - y)^2 + 2u^2\left\{\frac{1}{u}(b - ca) - g(y)\right\}^2 \\
&\quad \le \beta\left\{(a - y)^2 + \left[\frac{1}{u}(b - ca) - g(y)\right]^2\right\} \\
&\quad < \beta s^2 = \frac{r^2}{2} < r^2.
\end{aligned}$$

Here we have used the inequality $(v + w)^2 \le 2\big(v^2 + w^2\big)$. Thus the point $\big(y, ug(y) + cy\big)$ lies in the disc $D_r(a, b)$. Since $ug(y) + cy = f(y)$, it follows that $\big(y, f(y)\big)$ lies in $D_r(a, b)$. This shows that the graph of f is dense in $\mathbb{R}^2$. ∎

Here is an alternate proof of the denseness of the graph of an additive function which is not linear. Take $\alpha = f(1)$ and choose t such that $f(t) \neq \alpha t$. Let

$$E = \Big\{\big(x + yt, x\alpha + yf(t)\big) \ : \ x, y \in \mathbb{R}\Big\}.$$

Then $E \subset G(f)$. Consider the matrix

$$A = \begin{pmatrix} 1 & \alpha \\ t & f(t) \end{pmatrix}.$$

We observe that A is non-singular and hence is a homeomorphism of $\mathbb{R}^2$ on to itself. In particular A maps dense sets on to dense sets. The set $\{(x, y) : x, y \in \mathbb{Q}\}$ is a dense subset of $\mathbb{R}^2$ and its image under A is precisely E. Since E is contained in $G(f)$, it follows that $G(f)$ is dense $\mathbb{R}^2$.

Thus any additive function on $\mathbb{R}$ whose graph is not dense in $\mathbb{R}^2$ is necessarily linear.

We say a subset A of $\mathbb{R}^2$ is *closed* if whenever a sequence in A converges to some limit, that limit is also in A. Thus if $\langle (x_n, y_n) \rangle$ is a sequence in A, and if $x_n \to x$ and $y_n \to y$ as $n \to \infty$, then (x, y) is also in A. (This is not general definition of closed sets adopted in topological spaces, but it is sufficient to us here.) For example x-axis considered as a subset of $\mathbb{R}^2$ is closed. Interestingly the closedness of the graph of f will determine an additive function on $\mathbb{R}$.

Problem 4.6 let $f : \mathbb{R} \to \mathbb{R}$ be an additive function such that $G(f)$, the graph of f, is a closed subset of $\mathbb{R}^2$. Prove that there exists a constant c such that $f(x) = cx$, for all real x

Solution: Since f is additive, we have seen earlier that $f(r) = cr$ for all rationals r, where $c = f(1)$. Let us take any $x \in \mathbb{R}$ and choose a sequence $\langle r_n \rangle$ such that $r_n \to x$ as $n \to \infty$. But then $f(r_n) = cr_n$ and hence $f(r_n) \to cx$ as $n \to \infty$. Thus the sequence $\langle (r_n, f(r_n)) \rangle$ is a sequence in $G(f)$ and it converges to (x, cx). Since $G(f)$ is closed, it follows that (x, cx) is in $G(f)$. The definition of $G(f)$ shows that $f(x) = cx$.

Thus the closedness of $G(f)$ shows that $f(x) = cx$ for all real x. ∎

Here is another topological condition which implies linearity for an additive function. We have already defined the concept of an open disc in $\mathbb{R}^2$. We say a subset A of $\mathbb{R}^2$ is an *open set* if for each (a,b) in A there exists a $\delta > 0$ such that $D_\delta(a,b)$ is completely contained in A. For example each open disc is an open set in the above sense. Or $\{(x,y) \mid x > 0, y > 0\}$ is an open set. But x-axis is not an open set. We adopt the convention that $\emptyset$(*empty set*) is an open set. Obviously $\mathbb{R}^2$ itself is an open set. It is easy to check that an arbitrary union of open sets in $\mathbb{R}^2$ is again open. Similarly finite intersection of open sets is open. If we take $r_n = 1 + \frac{1}{n}$, then each $D_{r_n}(0,0)$ is an open set in $\mathbb{R}^2$. But $\cap_n D_{r_n}(0,0)$ is the set $\{(x,y) \mid x^2 + y^2 \leq 1\}$ which is not an open set; thus an infinite intersection of open sets may fail to be open.

We say a subset G of $\mathbb{R}^2$ is a G_δ set if G is a countable intersection of open sets; i.e., $G = \cap_{n=1}^{\infty} G_n$, where each G_n is an open set. For example, the set $\{(x,y) \mid$ at least one of x and y is irrational $\}$ is a G_δ set.

If f is an additive function on $\mathbb{R}$ such that $G = G(f)$ is a G_δ set in $\mathbb{R}^2$, then we can show that G is necessarily closed and hence f is linear. The proof uses more advanced ideas like Baire category theorem and we won't pursue this here. For a proof of this result see [2].

There is also measure theoretical condition(s) which would imply the linearity of an additive function. This condition is milder than continuity, but for an additive function this is enough to force continuity. The concepts get deeper and need more developments of basic material than what we have done so far. The interested readers are urged to look for these things in literature. For example see [1].

There are equations similar to (1) and they are called equations of Cauchy's type:

$$f(x+y) = f(x)f(y); \tag{4}$$

$$f(xy) = f(x) + f(y); \tag{5}$$

$$f(xy) = f(x)f(y). \tag{6}$$

Essentially equations (1),(4),(5) and (6) are questions of finding mappings from $\mathbb{R}$ to $\mathbb{R}$ which preserve certain algebraic structures. Equation (1) (i.e., Cauchy's equation) is to find all functions which preserve addition on $\mathbb{R}$. Similarly

(4) is a problem of determining all functions which take addition on $\mathbb{R}$ to multiplication on $\mathbb{R}$ and such interpretations can be given to the other two problems.

Such mappings are called homomorphisms. The set $\mathbb{R}$ has two structures: one is addition and other is multiplication. We know that $\mathbb{R}$ is a group with addition and $\mathbb{R}' = \mathbb{R}\setminus\{0\}$ is a group with multiplication. Equation (1) seeks to find all homomorphisms of $\mathbb{R}$ in to $\mathbb{R}$; (4) is to determine all homomorphisms of $\mathbb{R}$ in to $\mathbb{R}'$. Similarly (5) requires us to determine all homomorphisms from $\mathbb{R}'$ in to $\mathbb{R}$ and (6) is a problem of finding all homomorphisms of $\mathbb{R}'$ in to $\mathbb{R}'$.

We can follow the same method to solve equations (4), (5) and (6), with obvious modifications, as we have employed in the case of Cauchy's equation (1). We can go up to the stage of determining f on rational numbers without any further hypothesis. But we have to make additional assumptions like continuity on the functions f to get not too *weird* functions as solutions. The only continuous solutions of these equations are:

$$f(x) \equiv 0, \text{ and } f(x) = \exp(\alpha x) \quad \text{for equation (4);}$$

$$f(x) \equiv 0, \text{ and } f(x) = \alpha \log(x) \quad \text{for equation (5);}$$

$$f(x) \equiv 0, \text{ and } f(x) = x^{\alpha} \quad \text{for equation (6).}$$

There are some obvious generalisations of these problems. We consider a few of them here.

Problem 4.7 Find all functions $f, g, h : \mathbb{R} \to \mathbb{R}$ such that

$$f(x+y) = f(x)g(y) + h(y), \tag{1}$$

for all real numbers x, y.

Solution: This is a problem of determining three functions using a single equation. Let us see if we can reduce this to some familiar equation(s). Putting $x = 0$ in (1), we obtain

$$f(y) = f(0)g(y) + h(y). \tag{2}$$

Subtracting (2) from (1), we get $f(x+y)-f(y)=\big(f(x)-f(0)\big)g(y)$. Introducing $\phi(x)=f(x)-f(0)$, this reduces to

$$\phi(x+y)=\phi(x)g(y)+\phi(y), \tag{3}$$

where $\phi(0)=0$. Interchanging x and y, we also obtain

$$\phi(x+y)=\phi(y)g(x)+\phi(x). \tag{4}$$

Comparing (3) and (4), we are lead to the relation

$$\phi(x)\big[g(y)-1\big]=\phi(y)\big[g(x)-1\big]. \tag{5}$$

If $g(x)\equiv 1$, then (3) shows that ϕ satisfies Cauchy's equation and hence we have $\phi(x)=f_0(x)$, a solution of Cauchy's equation. In this case we can write all solutions of (1): $f(x)=f_0(x)+a$, $g(x)\equiv 1$ and $h(x)=f_0(x)$, for some constant a.

Suppose $g(x)\not\equiv 1$. Then we can find a real α such that $g(\alpha)\neq 1$. Putting $y=\alpha$ in (5), we can solve for $\phi(x)$:

$$\phi(x)=\frac{\phi(\alpha)}{g(\alpha)-1}\Big(g(x)-1\Big)=\beta\Big(g(x)-1\Big). \tag{6}$$

If $\beta=0$, then $\phi(x)=0$ for all x. In this case $f(x)=f(0)=a$, $g(x)$ arbitrary and $h(x)=a\big(1-g(x)\big)$ determine all solutions of (1). If $\beta\neq 0$, then putting (6) in (3), we obtain

$$g(x+y)=g(x)g(y).$$

In this case we know the general solution for g: $g(x)\equiv 0$ or $g(x)=\exp\big(f_0(x)\big)$ where f_0 satisfies Cauchy's equation. Setting $\gamma=a-\beta$, the general solution is

$$f(x)=\gamma,\quad g(x)\equiv 0,\ \text{and}\ h(x)=\gamma,$$

or

$$f(x)=\beta\exp\big(f_0(x)\big)+\gamma,\quad g(x)=\exp\big(f_0(x)\big),$$

$$h(x) = \gamma\big(1 - \exp(f_0(x))\big),$$

where f_0 satisfies $f_0(x+y) = f_0(x) + f_0(y)$. ∎

There is another important functional equation known as D'Alembert's equation. This first arose in the works of D'Alembert in his study of physical motion of strings. We shall consider this below and also a more general equation.

Problem 4.8 Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ satisfying the equation

$$f(x+y) + f(x-y) = 2f(x)f(y), \tag{1}$$

for all real numbers x, y.

Solution: Putting $y = 0$ in (1), we obtain $2f(x) = 2f(x)f(0)$. Thus $f(x) \equiv 0$ or $f(0) = 1$. In the latter case we also see after taking $x = 0$ in (1) that $f(-y) = f(y)$, so that f is an *even* function. Taking $x = ny$ in (1), we obtain

$$f\big((n+1)y\big) = 2f\big(y\big)f\big(ny\big) - f\big((n-1)y\big). \tag{2}$$

Taking $y = x$ in (1), we get $f(2x) + 1 = 2f(x)^2$. Setting $t = 2x$, we can write this in the form

$$f\left(\frac{t}{2}\right)^2 = \frac{f(t)+1}{2}. \tag{3}$$

We observe here that the *cosine* and *hyperbolic cosine* functions satisfy (2) and (3). It is our aim to show that these are the only continuous solutions.

Since $f(0) = 1$ and f is continuous, there is an $\alpha > 0$ such that $f(x) > 0$ in the interval $[-\alpha, \alpha]$. In particular, we can take α such that $f(\alpha) > 0$. Here we distinguish two cases: $0 < f(\alpha) \leq 1$ and $f(\alpha) > 1$.

Suppose $0 < f(\alpha) \leq 1$. Choose β such that $0 \leq \beta \leq \pi/2$ and such that $f(\alpha) = \cos\beta$. We show that for any $x = \frac{n}{2^m}\alpha$

where $n > 0, m \geq 0$ are integers, the relation

$$f(x) = \cos\left(\frac{\beta}{\alpha}x\right), \tag{4}$$

holds good. This is already true for $n = 1$ and $m = 0$, by our choice of β. The general statement may be proved by induction on m and n. Taking $t = \alpha$ in (3), we obtain

$$f\left(\frac{\alpha}{2}\right)^2 = \frac{f(\alpha)+1}{2} = \frac{\cos\beta + 1}{2},$$

since $f(\alpha) = \cos\beta$. But it is known that

$$\frac{\cos\beta + 1}{2} = \cos^2\left(\frac{\beta}{2}\right).$$

Since both $f(\alpha/2)$ and $\cos(\beta/2)$ are assumed to be positive, we conclude that

$$f\left(\frac{\alpha}{2}\right) = \cos\left(\frac{\beta}{2}\right),$$

which is (4) for $n = 1$ and $m = 1$. Suppose it is valid for $n = 1$ and some natural number m. Then using (3), we obtain

$$\begin{aligned} f\left(\frac{\alpha}{2^{m+1}}\right)^2 &= \frac{f\left(\frac{\alpha}{2^m}\right)+1}{2} \\ &= \frac{\cos\left(\frac{\beta}{2^m}\right)+1}{2} \\ &= \cos^2\left(\frac{\beta}{2^{m+1}}\right). \end{aligned}$$

The positivity assumptions on $f(\alpha/2^{m+1})$ and $\cos(\beta/2^{m+1})$ show that

$$f\left(\frac{\alpha}{2^{m+1}}\right) = \cos\left(\frac{\beta}{2^{m+1}}\right).$$

We have thus proved (4) for $n = 1$ and for every nonnegative integer m.

Obviously (4) is true for $n = 2$, for then $\frac{n}{2^m}\alpha = \frac{\alpha}{2^{m-1}}$, which reduces to the earlier case. Taking $n = 3$ and $y = \alpha/2^m$ in (2), we get

$$\begin{aligned} f\left(\frac{3}{2^m}\alpha\right) &= 2f\left(\frac{\alpha}{2^m}\right)f\left(\frac{\alpha}{2^{m-1}}\right) - f\left(\frac{\alpha}{2^m}\right) \\ &= 2\cos\left(\frac{\beta}{2^m}\right)\cos\left(\frac{\beta}{2^{m-1}}\right) - \cos\left(\frac{\beta}{2^m}\right) \\ &= \cos\left(\frac{3}{2^{m+1}}\beta\right). \end{aligned}$$

Thus (4) is true for $n = 3$ and $m \geq 0$. Suppose it is true for some n and every $m \geq 0$. Taking $y = \alpha/2^m$ in (2), we obtain

$$\begin{aligned} f\left(\frac{n+1}{2^m}\alpha\right) &= 2f\left(\frac{\alpha}{2^m}\right)f\left(n\frac{\alpha}{2^m}\right) - f\left(\frac{n-1}{2^m}\alpha\right) \\ &= 2\cos\left(\frac{\beta}{2^m}\right)\cos\left(\frac{n}{2^m}\beta\right) - \cos\left(\frac{n-1}{2^m}\beta\right) \\ &= \cos\left(\frac{n+1}{2^m}\beta\right) \end{aligned}$$

Thus we have proved (4) for $n+1$ and $m \geq 0$. We conclude that (4) is valid for every $n \geq 1$ and every $m \geq 0$.

Since the set of all numbers of the form $(n/2^m)\alpha$, $n \geq 1, m \geq 0$ is dense in $[0, \infty)$, it now follows that

$$f(x) = \cos\left(\frac{\beta}{\alpha}x\right), \tag{5}$$

for all $x \geq 0$. Since f is also an *even* function (5) is valid for all real x.

If $f(\alpha) > 1$, then we can find a positive real β such that $f(\alpha) = \cosh\beta$. The same analysis can be carried in this case

and it may be easily proved that

$$f(x) = \cosh\left(\frac{\beta}{\alpha}x\right).$$

■

There is also an obvious generalisation of D'Alembert's equation.

Problem 4.9 Find all continuous functions $f, g : \mathbb{R} \to \mathbb{R}$ such that

$$f(x+y) + f(x-y) = 2f(x)g(y). \tag{1}$$

Solution: We may assume $f(x) \not\equiv 0$; if $f(x) \equiv 0$, then we can choose g to be an arbitrary function. Suppose $f(a) \neq 0$. Put $x = a$ and replace y by $-y$ in (1) to get

$$2f(a)g(-y) = f(a-y) + f(a+y) = 2f(a)g(y).$$

Since $f(a) \neq 0$, it follows that $g(-y) = g(y)$ so that g is necessarily an *even* function. Taking $x = a$ and $y = 0$ in (1), we obtain $2f(a) = 2f(a)g(0)$ so that $g(0) = 1$. Thus $g(y) \not\equiv 0$. Introducing functions ϕ and ψ by

$$\phi(t) = \frac{f(t) + f(-t)}{2}, \quad \psi(t) = \frac{f(t) - f(-t)}{2},$$

we see that $f(x) = \phi(x) + \psi(x)$. Here ϕ is an *even* function, ψ is an *odd* function and they are respectively known as *even* and *odd* parts of f. Using these functions, we can write (1) in the form

$$\phi(x+y) + \phi(x-y) + \psi(x+y) + \psi(x-y) = 2\phi(x)g(y) + 2\psi(x)g(y). \tag{2}$$

Changing x to $-x$ and y to $-y$ and using the fact that g is an *even* function, we also obtain

$$\phi(x+y)+\phi(x-y)-\psi(x+y)-\psi(x-y) = 2\phi(x)g(y)-2\psi(x)g(y). \tag{3}$$

Adding (2) and (3), we get

$$\phi(x+y)+\phi(x-y)=2\phi(x)g(y). \tag{4}$$

Interchanging x and y in (4) and using the *even*-ness of ϕ, we obtain

$$\phi(x+y)+\phi(x-y)=2\phi(y)g(x). \tag{5}$$

Comparing (4) and (5), we are lead to the relation

$$\phi(x)g(y)=\phi(y)g(x). \tag{6}$$

Now taking $y=0$ in (6), we see that $\phi(x)=cg(x)$, where $c=\phi(0)$ (we know that $g(0)=1$).

If $c\neq 0$, then we substitute $\phi(x)=cg(x)$ in (4) and obtain

$$g(x+y)+g(x-y)=2g(x)g(y).$$

This is D'Alembert's equation and we have already determined all continuous solutions of this equation: $g(x)=\cos\alpha x$ and $g(x)=\cosh\alpha x$. (We have to avoid $g(x)\equiv 0$ since $g(0)=1$.) This gives $\phi(x)$ in the case $\phi(x)\not\equiv 0$.

If $\phi(x)\equiv 0\equiv\psi(x)$, then $f(x)\equiv 0$ and we can choose g arbitrarily. If $\phi(x)\not\equiv 0$ and $\psi(x)\equiv 0$, then we have $f(x)=\phi(x)=cg(x)$ and we have determined g in this case: $g(x)=\cos\alpha x$ or $g(x)=\cosh\alpha x$. Thus it remains to analyse the case $\psi(x)\not\equiv 0$.

Suppose $\psi(\alpha) \neq 0$ for some α. Subtracting (3) from (2), we obtain

$$\psi(x+y) + \psi(x-y) = 2\psi(x)g(y). \tag{7}$$

Now interchanging x and y in (7) and using the fact that ψ is an *odd* function, we get

$$\psi(x+y) - \psi(x-y) = 2\psi(y)g(x). \tag{8}$$

Subtraction of (8) from (7) leads to the relation

$$\psi(x-y) = \psi(x)g(y) - \psi(y)g(x). \tag{9}$$

If we set $y = \alpha$ in (9), we can solve for $g(x)$ to get

$$g(x) = \frac{g(\alpha)\psi(x) - \psi(x-\alpha)}{\psi(\alpha)}. \tag{10}$$

If we use (8), we finally get an equation for $g(x)$:

$$g(x+y) + g(x-y) = 2g(x)g(y), \tag{11}$$

which is D'Alembert's equation. We know that the continuous solutions of (11), with $g(0) = 1$, are $g(x) = \cos bx$ and $g(x) = \cosh bx$.

If $b = 0$, then $g(x) \equiv 1$ and putting this in (9) we obtain

$$\psi(x-y) = \psi(x) - \psi(y). \tag{12}$$

The continuous solutions of this equation are of the form $\psi(x) = \lambda x$. Thus we get $f(x) = \lambda x$ and $g(x) \equiv 1$.

Suppose $b \neq 0$. Then we have to solve the equations

$$\psi(x+y) + \psi(x-y) = 2\psi(x)\cos by, \tag{13}$$

$$\psi(x+y) + \psi(x-y) = 2\psi(x)\cosh by, \tag{14}$$

under the restriction that ψ is an *odd* function.

We consider the following more general problem of determining all *odd* functions f such that

$$f(x+y)+f(x-y)=2f(x)h(y), \tag{15}$$

where h is a given non constant *even* function. Choose r such that $h(x-2r) \not\equiv h(x)$. Let f_p be a particular solution of (15) and f a general solution. Choose constants A and B, not both *zero*, such that $Af_p(r)+Bf(r)=0$. Define $F(x)=Af_p(x)+Bf(x)$ so that $F(r)=0$. Interchanging x and y in (15), we see that

$$f(x+y)-f(x-y)=2f(y)h(x). \tag{16}$$

Subtracting (16) from (15), we get

$$f(x-y)=f(x)h(y)-f(y)h(x). \tag{17}$$

Since f_p is a particular solution of (15), it also satisfies (17). Thus F itself obey (17) and it follows that

$$F(x-y)=F(x)h(y)-F(y)h(x). \tag{18}$$

We show that $F(x) \equiv 0$. Suppose there exists β such that $F(\beta) \neq 0$. Putting $y=\beta$ in (18) and solving for g, we obtain

$$h(x)=\frac{h(\beta)F(x)-F(x-\beta)}{F(\beta)}. \tag{19}$$

Taking $x=\beta$ and $y=r$ in (18), we also get $F(\beta-r)=F(\beta)h(r)$ because $F(r)=0$. Taking $y=-r$ and using the fact that F is an *odd* function, h is an *even* function, we also get $F(x+r)=F(x)h(r)$. Thus we see that

$$F(\beta)=F(\beta-r+r)=F(\beta-r)h(r)=F(\beta)h(r)^2.$$

Since $F(\beta) \neq 0$, it follows that $h(r)^2=1$. Using this we obtain

$$F(x-2r)=F(x-r)h(r)=F(x)h(r)^2=F(x).$$

Now (19) shows that $h(x-2r) \equiv h(x)$, contradicting the choice of r. We conclude that $f(x) \equiv 0$. If $f_p(x) \not\equiv 0$, then $B \neq 0$ and hence $f(x) = -Af_p(x)/B$ determines the general solution of (15).

In the case of (13), we can take $f_p(x) = \sin \alpha x$ and we get $f(x) = d \sin \alpha x$ as the general solution of (13). Similarly, taking $f_p(x) = \sinh \alpha x$ as a particular solution of (14), we obtain $f(x) = d \sinh \alpha x$ as the general solution of (14).

The solutions of (1) are, therefore,

(a) $f(x) = c \cos \alpha x + d \sin \alpha x$, $g(x) = \cos \alpha x$,
(b) $f(x) = c \cosh \alpha x + d \sinh \alpha x$, $g(x) = \cosh \alpha x$,
(c) $f(x) = c + dx$, $g(x) \equiv 1$,
(d) $f(x) \equiv 0$, and g arbitrary. ∎

There is a class of functional equations satisfied by *sine* and *cosine* functions. For example we know that $\sin(x+y) = \sin x \cos y + \sin y \cos x$. If we set $g(x) = \sin x$ and $f(x) = \cos x$, then we can write the equation in the form

$$g(x+y) = g(x)f(y) + f(x)g(y).$$

Similarly, we can obtain three more relations

$$f(x+y) = f(x)f(y) - g(x)g(y),$$

$$g(x-y) = g(x)f(y) - g(y)f(x),$$

$$f(x-y) = f(x)f(y) + g(x)g(y).$$

What about the converse problem? Can we get *sine* and *cosine* back from any one of these equations? There are lot more continuous solutions in the case of first three equations. However the only continuous solutions of the fourth equation are $f(x) = \cos \alpha x$ and $g(x) = \sin \alpha x$.

Problem 4.10 Show that the only nontrivial continuous solutions of the functional equation

$$f(x-y) = f(x)f(y) + g(x)g(y) \tag{1}$$

for functions $f, g : \mathbb{R} \to \mathbb{R}$ are

$$f(x) = \cos \alpha x, \quad g(x) = \sin \alpha x.$$

Solution: We show that, we can reduce (1) to the familiar D'Alembert's equation. We assume that $f(x)$ is not identically a constant function. Otherwise the only possible solution is $f(x) = c$, $g(x) = \sqrt{c(1-c)}$, which exists under the restriction $0 \leq c \leq 1$. Interchanging x and y in (1), we obtain

$$f(y-x) = f(y)f(x) + g(y)g(x) = f(x-y).$$

Thus f is an *even* function. If $g(x)$ is also *even*, then changing y to $-y$ in (1) leads to

$$\begin{aligned} f(x+y) &= f(x)f(-y) + g(x)g(-y) \\ &= f(x)f(y) + g(x)g(y) = f(x-y), \end{aligned}$$

in view of the fact that both f and g are *even*. This forces f to be a constant function contrary to our assumption that f is not constant. Thus $g(x)$ is not an *even* function and hence cannot be constant.

Changing x to $-x$ and y to $-y$ in (1), we get

$$f(y-x) = f(-x)f(-y) + g(-x)g(-y),$$

which leads, using that f is an *even* function, to

$$f(x-y) = f(x)f(y) + g(-x)g(-y). \tag{2}$$

Comparing this with (1), we conclude that

$$g(x)g(y) = g(-x)g(-y). \tag{3}$$

In particular we obtain $g(x)^2 = g(-x)^2$ so that $g(-x) = \pm g(x)$. Since $g(x)$ is not constant, we can choose a such that

$g(a) \neq 0$. If $g(a) = g(-a)$, then taking $y = a$ in (3), we see that $g(x) = g(-x)$ for all x. This contradicts the observed fact that g is not even. Thus it follows that $g(-a) = -g(a)$ and this with (3) forces $g(-x) = -g(x)$. We conclude that $g(x)$ is necessarily an *odd* function. In particular $g(0) = 0$.

Taking $y = 0$ in (1), we obtain $f(x) = f(x)f(0)$. Since $f(x) \not\equiv 0$, we must have $f(0) = 1$. Putting $y = x$ in (1), we also see that

$$1 = f(x)^2 + g(x)^2.$$

Thus we have $|f(x)| \leq 1$ for all $x \in \mathbb{R}$. replacing y by $-y$ in the relation (1), we obtain

$$f(x+y) = f(x)f(y) - g(x)g(y). \tag{4}$$

Using (1) and (4), we finally get

$$f(x+y) + f(x-y) = 2f(x)f(y). \tag{5}$$

We have seen earlier that the non constant continuous solutions of (5) are $f(x) = \cos \alpha x$ and $f(x) = \cosh \alpha x$. However the condition $|f(x)| \leq 1$ throughout the domain entails us to choose only $f(x) = \cos \alpha x$.

Taking $y = a$ and $f(x) = \cos \alpha x$ in (1) and using $g(a) \neq 0$, we are lead to

$$g(x) = \frac{\cos \alpha(x-a) - \cos \alpha x \cos \alpha a}{g(a)} = \beta \sin \alpha x.$$

However using $f(x)^2 + g(x)^2 = 1$, we obtain $1 = 1 + (\beta^2 - 1)(\sin \alpha x)^2$. This forces $\beta^2 = 1$ and thus $g(x) = \pm \sin \alpha x$. If $g(x) = -\sin \alpha x$, then we can take $\delta = -\alpha$ and we obtain $f(x) = \cos \delta x$, $g(x) = \sin \delta x$. ∎

The following problems depend essentially on the solutions of Cauchy's problem and its variants.

Problem 4.11 Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ satisfying the equation

$$f(x+y) + f(xy) = f(x) + f(y) + f(xy+1). \qquad (1)$$

Solution: Observe that $f(1) = 0$. Put $F(x,y) = f(x) + f(y) - f(x+y)$. Then it is easy to check that

$$F(x+y,z) + F(x,y) = F(x,y+z) + F(y,z). \qquad (2)$$

From the given equation, we see that $F(x,y) = f(xy) - f(xy+1)$. Substituting this in (2), we obtain

$$\begin{aligned} &f(xz+yz) - f(xz+yz+1) + f(xy) - f(xy+1) \\ &\quad = f(xy+xz) - f(xy+xz+1) + f(yz) - f(yz+1). \end{aligned}$$

Substituting $y = 1/z$ in the above relation and introducing $x/y = u$, $xy = v$, we obtain

$$\begin{aligned} &f(u+1) - f(u+2) + f(v) - f(v+1) \\ &\qquad = f(u+v) - f(u+v+1) - f(2). \end{aligned} \qquad (3)$$

We have used here that $f(1) = 0$. We observe that u and v have same sign. Interchanging u and v, and comparing the resulting expression with (3), we see that

$$2f(u+1) - f(u+2) - f(u) = 2f(v+1) - f(v+2) - f(v).$$

Thus $2f(u+1) - f(u+2) - f(u) = c$, a constant. (Initially it looks as if $2f(u+1) - f(u+2) - f(u) = c_1$ for all $u \geq 0$ and $2f(u+1) - f(u+2) - f(u) = c_2$ for all $u \leq 0$. However we see that $c_1 = c_2$ taking $u = 0$.) It follows that $f(u+1) - f(u+2) = c + f(u) - f(u+1)$, where $c = -\big(f(2) + f(0)\big)$. Using this in (3), we obtain

$$\begin{aligned} &c + f(u) - f(u+1) + f(v) - f(v+1) \\ &\qquad = f(u+v) - f(u+v+1) - f(2). \end{aligned}$$

If we introduce $h(x) = f(x+1) - f(x) - \big(c + f(2)\big)$, then h satisfies the equation

$$h(u+v) = h(u) + h(v),$$

provided u and v are real numbers having the same sign. The only continuous solution of this equation is $h(x) = \alpha_1 x$ for all $x \geq 0$ and $h(x) = \alpha_2 x$ for all $x \leq 0$. We show that $\alpha_1 = \alpha_2$.

Let $\gamma = c + f(2)$. Then $f(x+1) - f(x) = \alpha_1 x + \gamma$ for $x \geq 0$ and $f(x+1) - f(x) = \alpha_2 x + \gamma$ for all $x \leq 0$. Hence $f(2) = \alpha_1 + \gamma$ (recall $f(1) = 0$) and $f(0) - f(-1) = -\alpha_2 + \gamma$. Taking $x = 2, y = -1$ in (1), we obtain $f(-2) = f(2) + 2f(-1)$. However the form of f for $x \leq 0$ shows that $f(-2) = f(-1) + 2\alpha_2 - \gamma$. Using this value we see that $-c = f(2) + f(0) = \alpha_2$. But we also have $f(2) = \alpha_1 + \gamma = \alpha_1 + c + f(2)$ showing that $-c = \alpha_1$. We conclude that $\alpha_1 = \alpha_2$.

Thus it follows that $f(x+1) - f(x) = \alpha x + \gamma$ for some constants α and γ. Substituting this in (1), we obtain

$$f(x+y) = f(x) + f(y) + \alpha xy + \gamma. \tag{4}$$

If we set $g(x) = f(x) - \gamma - \big(\alpha/2\big)x^2$, the g satisfies the relation

$$g(x+y) = g(x) + g(y).$$

Now the continuity of g forces that $g(x) = \beta x$, for some constant β. Thus we obtain

$$f(x) = \frac{\alpha}{2}x^2 + \beta x + \gamma.$$

Since we need $f(1) = 0$, these constants should also satisfy the condition $\big(\alpha/2\big) + \beta + \gamma = 0$.

Problem 4.12 Find all continuous functions $f, g, h : \mathbb{R} \to \mathbb{R}$ which satisfy the relation

$$f(x+y) + g(x-y) = 2h(x) + 2h(y), \tag{1}$$

for all $x, y \in \mathbb{R}$.

Solution: Setting $y = x$ in (1) and taking $g(0) = a$, we get

$$f(2x) = 4h(x) - a.$$

Now changing x to $x/2$, the relation transforms to

$$f(x) = 4h\left(\frac{x}{2}\right) - a. \tag{2}$$

Taking $y = 0$ in (1), we obtain a relation for $g(x)$:

$$g(x) = 2h(x) + 2b - 4h\left(\frac{x}{2}\right) + a, \tag{3}$$

where $h(0) = b$. Plugging (2) and (3) back in (1) and simplifying, the following relation emerges:

$$2\left[h\left(\frac{x+y}{2}\right) - h\left(\frac{x-y}{2}\right)\right] + h(x-y) + b = h(x) + h(y). \tag{4}$$

Introducing $H(x) = h(x) - b$, we see that $H(0) = 0$ and (4) can be written in the form

$$2\left[H\left(\frac{x+y}{2}\right) - H\left(\frac{x-y}{2}\right)\right] + H(x-y) = H(x) + H(y). \tag{5}$$

Changing x to $-x$ in (5), we obtain

$$2\left[H\left(\frac{-x+y}{2}\right)-H\left(\frac{-x-y}{2}\right)\right]+H(-x-y) = H(-x)+H(y). \tag{6}$$

Similarly changing y to $-y$ in (5), we also obtain,

$$2\left[H\left(\frac{x-y}{2}\right)-H\left(\frac{x+y}{2}\right)\right]+H(x+y)=H(x)+H(-y). \tag{7}$$

Now we introduce two functions $H_e(x)$ and $H_o(x)$, respectively known as *even* and *odd* part of $H(x)$, by

$$H_e(x)=\frac{H(x)+H(-x)}{2}, \quad H_o(x)=\frac{H(x)-H(-x)}{2}.$$

It is easy to see that $H_e(x)$ is an *even* function and $H_o(x)$ is an *odd* function. Adding (6) and (7), we get an equation for H_e:

$$2\left[H_e\left(\frac{x-y}{2}\right)-H_e\left(\frac{x+y}{2}\right)\right]+H_e(x+y)=H_e(x)+H_e(y). \tag{8}$$

If we change y to $-y$ in (8), we obtain

$$2\left[H_e\left(\frac{x+y}{2}\right)-H_e\left(\frac{x-y}{2}\right)\right]+H_e(x-y) = H_e(x)+H_e(-y). \tag{9}$$

Adding (8) and (9) and using $H_e(-y)=H_e(y)$, we are lead to the relation

$$H_e(x+y)+H_e(x-y)=2H_e(x)+2H_e(y). \tag{10}$$

Taking $x = y$ in (10) and using $H_e(0) = 0$, we obtain $H_e(2y) = 4H_e(y)$. We show that $H_e(ny) = n^2 H_e(y)$ for all natural numbers n and real numbers y. We have verified this for $n = 1$ and 2. Suppose this is true for all $k \leq m$, where $m \in \mathbb{N}$. Taking $x = my$ in (10), we see that

$$H_e\big((m+1)y\big) + H_e\big((m-1)y\big) = 2H_e(my) + 2H_e(y).$$

But $H_e\big((m-1)y\big) = (m-1)^2 H_e(y)$ and $H_e(my) = m^2 H_e(y)$ by induction hypothesis. It follows that

$$H_e\big((m+1)y\big) = \Big[2m^2 - (m-1)^2 + 2\Big] H_e(y) = (m+1)^2 H_e(y).$$

This proves our claim, by principle of induction, that $H_e(ny) = n^2 H_e(y)$ for all $n \in \mathbb{N}$ and $y \in \mathbb{R}$. Since $H_e(0) = 0$ and $H_e(-x) = H_e(x)$, we conclude that $H_e(ny) = n^2 H_e(y)$ for all integers n and real numbers y. In particular $H_e(n) = n^2 H_e(1) = \alpha n^2$ for all $n \in \mathbb{Z}$, where $\alpha = H_e(1)$ is a constant.

Let $r = p/q$ be a rational. Then we see that $\alpha p^2 = H_e(p) = H_e(qr) = q^2 H_e(r)$, so that $H_e(r) = \alpha r^2$ for all rationals r. Using the continuity of H_e, we conclude that $H_e(x) = \alpha x^2$ for all $x \in \mathbb{R}$.

Now changing x to $-x$ and y to $-y$ in (5), we obtain

$$2\left[H\left(\frac{-x-y}{2}\right) - H\left(\frac{-x-y}{2}\right)\right] + H(-x+y) = H(-x) + H(-y). \tag{11}$$

Subtraction of (11) from (5) gives

$$2\left[H_o\left(\frac{x+y}{2}\right) - H_o\left(\frac{x-y}{2}\right)\right] + H_o(x-y) = H_o(x) + H_o(y). \tag{12}$$

If we change y to $-y$ in (12), we get

$$2\left[H_o\left(\frac{x-y}{2}\right) - H_o\left(\frac{x+y}{2}\right)\right] + H_o(x+y) = H_o(x) + H_o(-y). \tag{13}$$

Addition of (12) and (13) leads to the relation

$$H_o(x+y) + H_o(x-y) = 2H_o(x), \tag{14}$$

since $H_o(-y) = -H_o(y)$. This reduces to Cauchy's equation and its continuous solution is given by $H_o(x) = \beta x$. Using $H(x) = H_e(x) + H_o(x)$, we get $H(x) = \alpha x^2 + \beta x$. From this it is easy to get

$$h(x) = \alpha x^2 + \beta x + b.$$

Putting this in (2) and (3), we get

$$f(x) = \alpha x^2 + 2\beta x + 4b - a, \quad g(x) = \alpha x^2 + a.$$

We check that

$$\begin{aligned} f(x+y) + g(x-y) &= \alpha(x+y)^2 + 2\beta(x+y) \\ &\quad + 4b - a + \alpha(x-y)^2 + a \\ &= 2h(x) + 2h(y), \end{aligned}$$

so that these functions f, g, h satisfy (1). ∎

Problem 4.13 Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x+y)f(x-y) = f(x)^2 - f(y)^2, \text{ for all } x, y \in \mathbb{R}. \tag{1}$$

Solution: Obviously $f(x) \equiv 0$ is a solution. We look for other solutions. Hence we may assume $f(a) \neq 0$ for some a. We also see that f is an *odd* function. Consider the function

$$g(x) = \frac{f(x+a) - f(x-a)}{2f(a)}. \tag{2}$$

We have

$$\begin{aligned} 2g(x)g(y) &= \frac{1}{2f(a)^2}\Big\{f(x+a) - f(x-a)\Big\} \\ &\qquad \Big\{f(y+a) - f(y-a)\Big\} \\ &= \frac{1}{2f(a)^2}\Big\{f(x+a)f(y+a) - f(x+a)f(y-a) \\ &\qquad - f(x-a)f(y+a) + f(x-a)f(y-a)\Big\}. \end{aligned}$$

We observe that

$$\begin{aligned} f(x+a)f(y+a) &= f\left(\frac{x+y}{2}+a\right)^2 - f\left(\frac{x-y}{2}\right)^2; \\ f(x+a)f(y-a) &= f\left(\frac{x+y}{2}\right)^2 - f\left(a+\frac{x-y}{2}\right)^2; \\ f(x-a)f(y+a) &= f\left(\frac{x+y}{2}\right)^2 - f\left(a-\frac{x-y}{2}\right)^2; \\ f(x-a)f(y-a) &= f\left(\frac{x+y}{2}-a\right)^2 - f\left(\frac{x-y}{2}\right)^2. \end{aligned}$$

On the other hand we also see that

$$\begin{aligned} f\left(\frac{x+y}{2}+a\right)^2 - f\left(\frac{x+y}{2}\right)^2 &= f(x+y+a)f(a); \\ f\left(a+\frac{x-y}{2}\right)^2 - f\left(\frac{x-y}{2}\right)^2 &= f(x-y+a)f(a); \\ f\left(\frac{x+y}{2}-a\right)^2 - f\left(\frac{x+y}{2}\right)^2 &= -f(x+y-a)f(a); \\ f\left(a-\frac{x-y}{2}\right)^2 - f\left(\frac{x-y}{2}\right)^2 &= -f(x-y-a)f(a). \end{aligned}$$

It follows that

$$\begin{aligned} 2g(x)g(y) &= \frac{1}{2f(a)}\Big\{f(x+y+a)-f(x+y-a) \\ &\qquad +f(x-y+a)-f(x-y-a)\Big\} \\ &= g(x+y)+g(x-y). \end{aligned} \tag{3}$$

Thus g satisfies D'Alembert's equation. A priori, it looks as if g depends on the point a we choose such that $f(a) \neq 0$. But this is only illusive and the right hand side of (2) does not depend on a. Let us take any y such that $f(y) \neq 0$. Setting $x+y=u$ and $x-y=v$ in (1), we obtain

$$f(u)f(v) = f\left(\frac{u+v}{2}\right)^2 - f\left(\frac{u-v}{2}\right)^2.$$

Thus we have

$$\begin{aligned} \frac{f(x+y)-f(x-y)}{2f(y)} &= \frac{f(x+y)f(a)-f(x-y)f(a)}{2f(a)f(y)} \\ &= \frac{1}{2f(a)f(y)}\left\{f\left(\frac{x+y+a}{2}\right)^2 - f\left(\frac{x+y-a}{2}\right)^2\right. \\ &\qquad \left. - f\left(\frac{x-y+a}{2}\right)^2 + f\left(\frac{x-y-a}{2}\right)^2\right\} \\ &= \frac{1}{2f(a)f(y)}\left\{f\left(\frac{x+a+y}{2}\right)^2 - f\left(\frac{x+a-y}{2}\right)^2\right. \\ &\qquad \left. - f\left(\frac{x-a+y}{2}\right)^2 + f\left(\frac{x-a-y}{2}\right)^2\right\} \\ &= \frac{f(x+a)f(y)-f(x-a)f(y)}{2f(a)f(y)} \\ &= \frac{f(x+a)-f(x-a)}{2f(a)} = g(x). \end{aligned}$$

This shows that g does not depend on a; we can use any y such that $f(y) \neq 0$. We thus obtain

$$f(x+y) - f(x-y) = 2g(x)f(y), \tag{4}$$

where g satisfies the relation

$$g(x+y) + g(x-y) = 2g(x)g(y). \tag{5}$$

We have to solve these under the conditions that f, g are continuous and $f(x) \not\equiv 0$.

Taking $y = 0$ in (5), we obtain $g(x) = g(x)g(0)$, valid for all x. If $g(x) = 0$ for all x, then the substitution $y = 0$ in (4) gives $f(2x) = f(0)$ showing that f is constant. But it is easy to see that (1) has no constant solutions except $f(x) \equiv 0$. Thus $g(x)$ cannot be identically 0 and hence $g(0) = 1$. Taking $x = 0$ in (4), we see that $f(y) - f(-y) = 2f(y)$ which shows that f is an *odd* function. Thus we can write (4) in the form

$$f(x+y) + f(x-y) = 2f(x)g(y). \tag{6}$$

We have solved this problem earlier (see problem 4.9). Its solutions are

(a) $f(x) = A\cos\alpha x + B\sin\alpha x$, $\quad g(x) = \cos\alpha x$;
(b) $f(x) = A\cosh\alpha x + B\sinh\alpha x$, $\quad g(x) = \cosh\alpha x$;
(c) $f(x) = A + Bx$, $\quad g(x) \equiv 1$;
(d) $f(x) \equiv 0$, and g arbitrary.

Since f is an *odd* function, it follows that $A = 0$ in all these solutions. Thus the continuous solutions of (1) are

$$f(x) \equiv 0,\ f(x) = Bx,\ f(x) = B\sin\alpha x,\ f(x) = B\sinh\alpha x.$$

It is easy to check that these are indeed the solutions. ■

Problem 4.14 Show that the continuous solutions of the equation

$$f(x+y) - f(x-y) = 2h(x)h(y), \tag{1}$$

where f, g are functions from $\mathbb{R}$ to $\mathbb{R}$, are given by

$$\begin{aligned} &h(x) \equiv 0, && f(x) \equiv k; \\ &h(x) = \alpha x, && f(x) = \left(\frac{\alpha^2}{2}\right)x^2 + k; \\ &h(x) = \alpha \sin \beta x, && f(x) = 2\alpha^2 \sin^2\left(\frac{\beta x}{2}\right) + k; \\ &h(x) = \alpha \sinh \beta x, && f(x) = 2\alpha^2 \sinh^2\left(\frac{\beta x}{2}\right) + k. \end{aligned}$$

Solution: Obviously $h(x) \equiv 0$, $f(x) \equiv k$ give one possible solution. We may assume that $h(x) \not\equiv 0$. Replacing y by $-y$ in (1), we get

$$f(x-y) - f(x+y) = 2h(x)h(-y). \tag{2}$$

The equations (1) and (2) lead to $h(x)h(-y) = -h(x)h(y)$. Since $h(x) \not\equiv 0$, it follows that $h(-y) = -h(y)$ so that h is an *odd* function. In particular $h(0) = 0$. Taking $x = 0$ in (1), we also obtain $f(-y) = f(y)$ so that f is an *even* function. Setting $y = x$ in (1) and changing x to $x/2$, we obtain

$$f(x) = 2h\left(\frac{x}{2}\right)^2 + f(0). \tag{3}$$

If we use this in (1), we get an equation only in h:

$$h\left(\frac{x+y}{2}\right)^2 - h\left(\frac{x-y}{2}\right)^2 = h(x)h(y).$$

Now using the change of variables $\dfrac{x+y}{2} = u$ and $\dfrac{x-y}{2} = v$, we finally obtain an equation

$$h(u)^2 - h(v)^2 = h(u+v)h(u-v). \tag{4}$$

This is same as problem 4.13 above. We have obtained there the continuous solutions:

$$h(x) = \alpha x; \quad h(x) = \alpha \sin \beta x; \quad h(x) = \alpha \sinh \beta x.$$

If we use these in (3), we obtain the values of f. ■

Problem 4.15 Find all functions $f, g : \mathbb{R} \to \mathbb{R}$ which satisfy the equation

$$g(x) - g(y) = (x - y)f(x + y) + (x + y)f(x - y), \qquad (1)$$

for all real numbers x, y.

Solution: Taking $y = 0$ in (1), we get

$$g(x) = 2xf(x) + \beta, \qquad (2)$$

where $\beta = g(0)$. Thus we get an equation involving only f:

$$2xf(x) - 2yf(y) = (x - y)f(x + y) + (x + y)f(x - y). \qquad (3)$$

A solution of (3) gives $g(x)$ in view of (2) and hence completely solves (1). Taking $x = y$ in (1), we see that $f(0) = 0$. Now $x = 0$ in (1) gives

$$g(y) = yf(y) - yf(-y) + \beta. \qquad (4)$$

Comparing (2) and (4), we obtain $f(-x) = -f(x)$ for all $x \neq 0$, so that $f(x)$ is an *odd* function. Replacing y by $y + z$ in (3), we obtain

$$\begin{aligned} &2xf(x) - 2(y + z)f(y + z) \\ &\quad = (x - y - z)f(x + y + z) + (x + y + z)f(x - y - z). \end{aligned} \qquad (5)$$

Changing x to $x+y$ and y to z, we obtain

$$\begin{aligned}&2(x+y)f(x+y)-2zf(z)\\&\quad=(x+y-z)f(x+y+z)+(x+y+z)f(x+y-z).\end{aligned}\tag{6}$$

Adding (5) and (6), we have

$$\begin{aligned}&2xf(x)-2zf(z)+2(x+y)f(x+y)-2(y+z)f(y+z)\\&\qquad=(x+y+z)\big(f(x-y-z)+f(x+y-z)\big)\\&\qquad\qquad+2(x-z)f(x+y+z).\end{aligned}\tag{7}$$

Using (3) in (7), we obtain

$$\begin{aligned}&(x-z)f(x+z)+(x+z)f(x-z)\\&\quad+(x-z)f(x+2y+z)+(x+2y+z)f(x-z)\\&\qquad=(x+y+z)\big(f(x-y-z)+f(x+y-z)\big)\\&\qquad\qquad+2(x-z)f(x+y+z).\end{aligned}\tag{8}$$

Taking $z=-x$ in (8), we get

$$2xf(2y)+2yf(2x)=4xf(y)+y\big(f(2x-y)+f(2x+y)\big).$$

Introducing $u=2x$, $v=y$, we may write this in the form

$$\frac{u}{v}f(2v)+2f(u)-\frac{2u}{v}f(v)=f(u-v)+f(u+v),\tag{9}$$

for all $v\neq 0$. Interchanging u and v, we also obtain

$$\frac{v}{u}f(2u)+2f(v)-\frac{2v}{u}f(u)=f(u+v)-f(u-v),\tag{10}$$

where we have used the *odd* property of $f(x)$. Adding (9) and (10), we get

$$\begin{aligned}&f(u+v)-f(u)-f(v)\\&\qquad=\frac{u}{2v}\big(f(2v)-2f(v)\big)+\frac{v}{2u}\big(f(2u)-2f(u)\big),\end{aligned}\tag{11}$$

for all $u, v \neq 0$. Introducing $h(x) = \big(f(2x) - 2f(x)\big)/2x$, for $x \neq 0$, (11) takes the form

$$f(u+v) - f(u) - f(v) = uh(v) + vh(u), \tag{12}$$

for all $u, v \neq 0$. The function $H(u,v) = f(u+v) - f(u) - f(v)$ satisfies the relation

$$H(u+v, w) + H(u, v) = H(u, v+w) + H(v, w). \tag{13}$$

Using this in (12), we see that

$$w\big(h(u+v) - h(u) - h(v)\big) = u\big(h(v+w) - h(v) - h(w)\big). \tag{14}$$

Taking $w = v$ in (14), we get

$$v\big(h(u+v) - h(u) - h(v)\big) = u\big(h(2v) - 2h(v)\big). \tag{15}$$

Interchanging u and v, we also get

$$u\big(h(u+v) - h(u) - h(v)\big) = v\big(h(2u) - 2h(u)\big). \tag{16}$$

Comparing (15) and (16), we have

$$u^2\big(h(2v) - 2h(v)\big) = v^2\big(h(2u) - 2h(u)\big).$$

This shows that

$$h(2u) - 2h(u) = 6\alpha u^2, \tag{17}$$

for all $u \neq 0$. Substituting (17) in (16), we get

$$h(u+v) - h(u) - h(v) = 6\alpha uv, \tag{18}$$

for all $u, v \neq 0$. Introducing $\varphi(x) = h(x) - 3\alpha x^2$, the function $\varphi(x)$ satisfies

$$\varphi(x+y) = \varphi(x) + \varphi(y),$$

for all $x, y, x+y \neq 0$. Taking $u \neq 0, 1$, we see that

$$\begin{aligned} \varphi(1) = \varphi(u+1-u) &= \varphi(u) + \varphi(1-u) \\ &= \varphi(u) + \varphi(1) + \varphi(-u). \end{aligned}$$

This shows that $\varphi(-u) = -\varphi(u)$ for all $u \neq 0, 1$. Using $\varphi(2) = 2\varphi(1)$, $\varphi(-2) = 2\varphi(-1)$, we also get $\varphi(-1) = -\varphi(1)$. Thus $\varphi(-u) = -\varphi(u)$ for all $u \neq 0$. Using this in the definition of $\varphi(x)$, we obtain $h(x) = 3\alpha x^2$ for all $x \neq 0$. Putting this in (12), we get

$$f(u+v) - f(u) - f(v) = 3\alpha u^2 v + 3\alpha u v^2,$$

for all $u, v \neq 0$. Putting $\psi(x) = f(x) - \alpha x^3$, the function $\psi(x)$ satisfies

$$\psi(x+y) = \psi(x) + \psi(y),$$

for all $x, y \neq 0$. Since $f(0) = 0$, this is also valid for $x = 0$ and $y = 0$ as well. Thus

$$f(x) = \alpha x^3 + \psi(x),$$

for some additive function $\psi(x)$. Finally

$$g(x) = 2\alpha x^4 + 2x\psi(x) + \beta.$$

∎

There are several nice applications of Cauchy's equations and D,Alemberts' equations. We consider here two of them.

Problem 4.16 Suppose $F(x, y)$ denote the area of a rectangle with sides x and y. Assuming $F(x, y)$ is a continuous function of x, y and linear in x and y, prove that $F(x, y) = \lambda xy$.

Solution: The conclusion of the problem is that the area of a rectangle is precisely what we itutively expect: product of its length and breadth. Note that $\lambda = 1$ after suitable normalisation. Thus if we assume that the area obeys conditions which we expect of it, then it is simply the product of the two sides for a rectangle. The linearity part says that

$$\begin{aligned} F(x_1 + x_2, y) &= F(x_1, y) + F(x_2, y), \\ F(x, y_1 + y_2) &= F(x, y_1) + F(x, y_2). \end{aligned}$$

Moreover, $F(x, y) > 0$ if $x > 0$ and $y > 0$.

Define for a fixed y, the function $f_y : \mathbb{R} \to \mathbb{R}$ by $F(x, y) = f_y(x)$. Then f_y is a continuous function for each y and

$$f_y(x_1 + x_2) = f_y(x_1) + f_y(x_2),$$

for all non-negative real numbers x_1 and x_2. It follows that

$$f_y(x) = h(y)x,$$

for all $x \geq 0$. Here $h(y)$ depends on y, but it is constant for a fixed y. Now using the linearity in y, we get

$$\begin{aligned} h(y_1 + y_2)x = f_{y_1+y_2}(x) &= F(x, y_1 + y_2) \\ &= F(x, y_1) + F(x, y_2) \\ &= f_{y_1}(x) + f_{y_2}(x) \\ &= \big(h(y_1) + h(y_2)\big)x, \end{aligned}$$

for all $x \geq 0$ and $y_1, y_2 \geq 0$. Thus $h(y)$ satisfies the equation

$$h(y_1 + y_2) = h(y_1) + h(y_2),$$

for all $y_1, y_2 \geq 0$. The continuity of $F(x, y)$ in y implies that $h(y)$ is a continuous function on the set of all non-negative real numbers. Again, the solution of Cauchy's equation implies that $h(y) = \lambda y$ for some constant λ. Combining h with f_y, we see that

$$F(x, y) = \lambda xy,$$

for all non-negative real numbers x and y.

The second application is for determining scalar(dot) and vector(cross) products of two vectors in a three dimensional space $\mathbb{R}^3$. Given two vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}^3$, their scalar and vector products are formally introduced by

$$\begin{aligned} \mathbf{a}\cdot\mathbf{b} &= |\mathbf{a}|\,|\mathbf{b}|\cos\angle(\mathbf{a},\mathbf{b}); \\ \mathbf{a}\times\mathbf{b} &= |\mathbf{a}|\,|\mathbf{b}|\sin\angle(\mathbf{a},\mathbf{b})\mathbf{e}' \end{aligned}$$

where $\mathbf{e}'$ denotes the unit vector perpendicular to $\mathbf{a}$ and $\mathbf{b}$. We wonder what could be the basis for such a definition. Note that the only relevant property of these products are the distributivity with respect to addition:

$$\begin{aligned} (\mathbf{a}+\mathbf{b})\cdot\mathbf{c} &= \mathbf{a}\cdot\mathbf{c}+\mathbf{b}\cdot\mathbf{c}; \\ (\mathbf{a}+\mathbf{b})\times\mathbf{c} &= \mathbf{a}\times\mathbf{c}+\mathbf{b}\times\mathbf{c}. \end{aligned}$$

However we can construct many products which obey distributivity. Example: if $\mathbf{a}=(a_1,a_2,a_3)$ and $\mathbf{b}=(b_1,b_2,b_3)$, we can consider

$$\mathbf{a}\bullet\mathbf{b} = a_1b_1 - a_2b_2 - a_3b_3.$$

What additional properties uniquely determine these products? One important property is that *there is no distinct direction* in the space. Thus the scalar product must be*invariant* under rotation and the vector product must undergo *the same rotation.* This along with linearity turns out to be the crucial deciding property of the scalar and vector products.

Problem 4.17 Suppose

(i) there is no *distinguished* direction in the space; i.e., the scalar product is invariant under rotation and the vector product undergoes the same rotation;

(ii) both the products are distributive on the right; i.e.,

$$(\mathbf{a}+\mathbf{b})\cdot\mathbf{c} = \mathbf{a}\cdot\mathbf{c}+\mathbf{b}\cdot\mathbf{c},$$
$$(\mathbf{a}+\mathbf{b})\times\mathbf{c} = \mathbf{a}\times\mathbf{c}+\mathbf{b}\times\mathbf{c},$$

for any three vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$;

(iii) both products are homogeneous; i.e.,

$$\begin{aligned}(\lambda\mathbf{a})\cdot\mathbf{b} = \lambda(\mathbf{a}\cdot\mathbf{b}) &= \mathbf{a}\cdot(\lambda\mathbf{b}),\\ (\lambda\mathbf{a})\times\mathbf{b} = \lambda(\mathbf{a}\times\mathbf{b}) &= \mathbf{a}\times(\lambda\mathbf{b}),\end{aligned}$$

for any two vectors $\mathbf{a}$ and $\mathbf{b}$, and scalar λ. Prove that

$$\begin{aligned}\mathbf{a}\cdot\mathbf{b} &= |\mathbf{a}|\,|\mathbf{b}|\cos\angle(\mathbf{a},\mathbf{b});\\ \mathbf{a}\times\mathbf{b} &= |\mathbf{a}|\,|\mathbf{b}|\sin\angle(\mathbf{a},\mathbf{b})\mathbf{e}'\end{aligned}$$

where $\mathbf{e}'$ denotes the unit vector perpendicular to $\mathbf{a}$ and $\mathbf{b}$.

Solution: We prove this in several steps.

(1) Suppose $\mathbf{a}$ and $\mathbf{b}$ are two vectors which are perpendicular to each other. We show that $\mathbf{a}\cdot\mathbf{b}=0$. Since we can write $\mathbf{a}=|\mathbf{a}|\mathbf{e}_1$ and $\mathbf{b}=|\mathbf{b}|\mathbf{e}_2$ for some unit vectors $\mathbf{e}_1$ and $\mathbf{e}_2$, property (iii) shows that it suffices to prove $\mathbf{e}_1\cdot\mathbf{e}_2=0$ for any two unit vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ which are perpendicular to each other. Suppose we rotate $\mathbf{e}_1$ by an angle π around the axis of $\mathbf{e}_2$. Then the pair $(\mathbf{e}_1,\mathbf{e}_2)$ gets transformed to the pair $(-\mathbf{e}_1,\mathbf{e}_2)$. Using the properties (ii) and (iii) for scalar product, we get

$$\mathbf{e}_1\cdot\mathbf{e}_2 = (-\mathbf{e}_1)\cdot\mathbf{e}_2 = -(\mathbf{e}_1\cdot\mathbf{e}_2).$$

This shows that $\mathbf{e}_1\cdot\mathbf{e}_2=0$.

(2) Suppose $\mathbf{a}$ and $\mathbf{b}$ are two vectors which are parallel to each other. Then $\mathbf{a} \times \mathbf{b} = 0$. Note that $\mathbf{a} = |\mathbf{a}|\mathbf{e}$ and $\mathbf{b} = |\mathbf{b}|\mathbf{e}$ for some unit vector $\mathbf{e}$ since $\mathbf{a}$ and $\mathbf{b}$ are parallel to each other. Hence it suffices to prove that $\mathbf{e} \times \mathbf{e} = 0$ for any unit vector $\mathbf{e}$. Now the assumption (i) implies that $\mathbf{e} \times \mathbf{e}$ is a vector only in the direction of $\mathbf{e}$ (or $-\mathbf{e}$) and the absolute value of $\mathbf{e} \times \mathbf{e}$ is independent of the direction of $\mathbf{e}$. Hence $\mathbf{e} \times \mathbf{e} = \lambda \mathbf{e}$ for some scalar λ. By rotating $\mathbf{e}$ by an angle π, it is transformed to $-\mathbf{e}$. Now (i) and (iii) give

$$-\lambda \mathbf{e} = \lambda(-\mathbf{e}) = (-\mathbf{e}) \times (-\mathbf{e}) = \mathbf{e} \times \mathbf{e} = \lambda \mathbf{e}.$$

Thus $\lambda = 0$ and this gives $\mathbf{e} \times \mathbf{e} = 0$.

(3) Let $\mathbf{e}$ be the unit vector orthogonal to the plane spanned by $\mathbf{a}$ and $\mathbf{b}$, and let $\mathbf{c} = \mathbf{a} \times \mathbf{b}$. Then the vectors $\mathbf{a}, \mathbf{b}, \mathbf{e}$ span $\mathbb{R}^3$ and hence we may write

$$\mathbf{c} = \lambda \mathbf{a} + \mu \mathbf{b} + \nu \mathbf{e},$$

for some scalars λ, μ and ν. Now a rotation by an angle π around the axis of $\mathbf{e}$ transforms the system $(\mathbf{a}, \mathbf{b}, \mathbf{e})$ to $(-\mathbf{a}, -\mathbf{b}, \mathbf{e})$. The assumptions (i) and (iii) give

$$\begin{aligned} \lambda \mathbf{a} + \mu \mathbf{b} + \nu \mathbf{e} &= \mathbf{c} = \mathbf{a} \times \mathbf{b} \\ &= (-\mathbf{a}) \times (-\mathbf{b}) = \lambda(-\mathbf{a}) + \mu(-\mathbf{b}) + \nu \mathbf{e}. \end{aligned}$$

It follows that $\lambda = \mu = 0$. Thus we get

$$\mathbf{a} \times \mathbf{b} = \nu \mathbf{e},$$

where $\mathbf{e}$ is a unit vector perpendicular to $\mathbf{a}$ and $\mathbf{b}$.

(4) Now we try to get a functional equation involving these products. Consider a unit vector $\mathbf{e}$. Let $\mathbf{e}_{\varphi+\psi}$, $\mathbf{e}_{\varphi-\psi}$

and $\mathbf{e}_\varphi$ be unit vectors coplanar with $\mathbf{e}$ and making angles $\varphi+\psi$, $\varphi-\psi$ and φ respectively with $\mathbf{e}$. Using the vector addition we have

$$\mathbf{e}_{\varphi+\psi}+\mathbf{e}_{\varphi-\psi}=2\mathbf{e}_\varphi\cos\psi.$$

The assumption (ii) gives

$$\begin{aligned}(\mathbf{e}_{\varphi+\psi}+\mathbf{e}_{\varphi-\psi})\cdot\mathbf{e} &= \mathbf{e}_{\varphi+\psi}\cdot\mathbf{e}+\mathbf{e}_{\varphi-\psi}\cdot\mathbf{e},\\ (\mathbf{e}_{\varphi+\psi}+\mathbf{e}_{\varphi-\psi})\times\mathbf{e} &= \mathbf{e}_{\varphi+\psi}\times\mathbf{e}+\mathbf{e}_{\varphi-\psi}\times\mathbf{e}.\end{aligned}$$

Let us write $\mathbf{e}_\varphi\cdot\mathbf{e}=f(\varphi)$ and $\mathbf{e}_\varphi\times\mathbf{e}=g(\varphi)\mathbf{e}'$, where $\mathbf{e}'$ is a unit vector perpendicular to $\mathbf{e}$ and $\mathbf{e}_\varphi$ such that $(\mathbf{e}_\varphi,\mathbf{e},\mathbf{e}')$ form a right-handed system. Note that $\mathbf{e}_{\varphi+\psi}\times\mathbf{e}$, $\mathbf{e}_{\varphi-\psi}\times\mathbf{e}$ and $\mathbf{e}_\varphi\times\mathbf{e}$ are parallel vectors. Thus both f and g satisfy the same equation:

$$\begin{aligned}f(\varphi+\psi)+f(\varphi-\psi) &= 2f(\varphi)\cos\psi,\\ g(\varphi+\psi)+g(\varphi-\psi) &= 2g(\varphi)\cos\psi.\end{aligned}$$

Here f and g are continuous functions. These are D'Alembert's equations. The only solution for f is

$$f(x)=\lambda\cos x+\mu\sin x,$$

for some constants λ and μ. But we know that $\mathbf{e}_1\cdot\mathbf{e}_2=0$ if $\mathbf{e}_1$ and $\mathbf{e}_2$ are perpendicular to each other. Thus $f(\pi/2)=0$ and we get

$$f(\varphi)=\lambda\cos\varphi.$$

If we take any two vectors $\mathbf{a}$ and $\mathbf{b}$ with angle φ, we can write

$$\mathbf{a}=|\mathbf{a}|\mathbf{e}_\varphi,\quad \mathbf{b}=|\mathbf{b}|\mathbf{e},$$

for some unit vectors $\mathbf{e}$ and $\mathbf{e}_\varphi$. Using property (iii) we get

$$\mathbf{a}\cdot\mathbf{b}=|\mathbf{a}||\mathbf{b}|\mathbf{e}_\varphi\cdot\mathbf{e}=\lambda|\mathbf{a}||\mathbf{b}|\cos\phi.$$

We also get for g,

$$g(x) = \lambda \cos x + \mu \sin x,$$

and $g(0) = 0$ since $\mathbf{e} \times \mathbf{e} = 0$. Hence $g(\varphi) = \mu \sin \varphi$. This leads to

$$\mathbf{a} \times \mathbf{b} = \mu |\mathbf{a}||\mathbf{b}| \sin \varphi \, \mathbf{e}',$$

where φ is the angle between $\mathbf{a}$ and $\mathbf{b}$; and $\mathbf{e}'$ is the unit vector perpendicular to $\mathbf{a}$ and $\mathbf{b}$. We can take a proper normalisation in which $\lambda = \mu = 1$, and we get the standard scalar and vector products.

∎

Another problem is pertaining to the *affine* group of real numbers. Suppose we consider the set of all affine functions on $\mathbb{R}$;

$$A = \{ f(x) = ax + b \ : \ a \neq 0 \}.$$

Then A is a group under composition of mappings. We want to find all group endomorphisms of A into itself.

Problem 4.18 Find all functions $F, H : (\mathbb{R} \setminus \{0\}) \times \mathbb{R} \to \mathbb{R}$ which satisfy the equations

$$F(xu, xv + y) = F(x, y)F(u, v), \tag{1}$$

$$G(xu, xv + y) = F(x, y)H(u, v) + H(x, y), \tag{2}$$

for all $x \neq 0$, $u \neq 0$, $y, v \in \mathbb{R}$; here $F(x, y) \neq 0$ for any pair (x, y).

Solution: Let us introduce $F(x, 0) = f(x)$ and $F(1, y) = g(y)$. Note that $f(x) \neq 0$ and $g(y) \neq 0$ for any x and y. Taking $y = v = 0$ in (1), we get

$$f(xu) = f(x)f(u). \tag{3}$$

Similarly, $x = u = 1$ in (1) gives

$$g(v + y) = g(y)g(v). \tag{4}$$

If we know f and g, we can construct back F; setting $x = 1$, $v = 0$, $u = s$, $y = t$ in (1), we have

$$F(s, t) = F(1, t)F(s, 0) = g(t)f(s), \tag{5}$$

so that $F(s, t)$ is determined by $g(t)$ and $f(s)$. Again, taking $u = 1$, $y = 0$, $x = s$, $v = t$ in (1), we obtain

$$F(s, st) = F(s, 0)F(1, t) = F(s, t).$$

Thus (5) implies

$$f(s)g(t) = f(s)g(st).$$

Since $f(s) \neq 0$ for any s, it follows that

$$g(st) = g(t),$$

for all $s \neq 0$. Taking $t = 1$, we obtain $g(s) = g(1) = \lambda \neq 0$ for all $s \neq 0$. Putting this in (4), we get $\lambda = 1$. Observe that (4) further implies $g(0) = 1$. We conclude that $g(t) = 1$ for all t. Hence (5) gives

$$F(s, t) = f(s), \tag{6}$$

for all $s \neq 0$ and t. Using this in (2), we obtain

$$H(xu, xv + y) = f(x)H(u, v) + H(x, y). \tag{7}$$

Here we consider two cases.

Case 1. Suppose $f(x) \equiv 1$. Then (7) reduces to

$$H(xu, xv + y) = H(u, v) + H(x, y).$$

Consider $H'(x,y) = e^{H(x,y)}$. Then $H'(x,y) \neq 0$ and H' satisfies the equation

$$H'(xu, xv+y) = H'(x,y)H'(u,v),$$

which is same as (1). Thus $H'(x,y) = f'(x)$, where f' satisfies the relation

$$f'(xu) = f'(x)f'(u).$$

We hence obtain the solution:

$$F(x,y) \equiv 1, \quad H(x,y) = \ln f'(x),$$

where f' is a solution of the equation $f'(xu) = f'(x)f'(u)$ with $f'(x) > 0$ for all x.

Case 2. Suppose $f(x) \not\equiv 1$, so that there is some x_0 such that $f(x_0) \neq 1$. Taking $y = v = 0$ in (7), we have

$$H(xu, 0) = f(x)H(u,0) + H(x,0).$$

The left side is symmetric in x and u. Thus we get

$$f(x)H(u,0) + H(x,0) = f(u)H(x,0) + H(u,0).$$

Taking $u = x_0$ and using $f(x_0) \neq 1$, we solve for $H(x,0)$:

$$H(x,0) = \lambda\big(f(x) - 1\big), \tag{8}$$

where

$$\lambda = \frac{H(x_0,0)}{f(x_0) - 1}.$$

Taking $x = 1$ in (3) and using $f(u) \neq 0$, we see that $f(1) = 1$. Setting $x = u = 1$ in (7), we get

$$H(1, v+y) = H(1,v) + H(1,y).$$

Thus $H(1,v) = h(v)$ satisfies the equation

$$h(v+y) = h(v) + h(y), \tag{9}$$

for all v, y. We can construct back $H(x, y)$ by (8) and (9). Putting $x = 1$, $v = 0$, $u = s$, $y = t$ in (7), we obtain

$$H(s,t) = f(1)H(s,0) + H(1,t) = \lambda\big(f(s) - 1\big) + h(t). \quad (10)$$

Taking $u = 1$, $y = 0$, $x = s$, $v = t$ in (7), we also obtain

$$H(s,st) = f(s)H(1,t) + H(s,0) = f(s)h(t) + \lambda\big(f(s) - 1\big). \quad (11)$$

Comparing (10) and (11), we get

$$h(st) = f(s)h(t). \quad (12)$$

Taking $h(1) = \alpha$, we see that (12) gives $h(s) = \alpha f(s)$, for all $s \neq 0$. If $\alpha = 0$, then $h(s) \equiv 0$. (Note that $h(0) = 0$ from (9).) In this case we get the solution

$$F(x,y) = f(x), \quad H(x,y) = \lambda\big(f(x) - 1\big),$$

where $f(xy) = f(x)f(y)$, $f(x) \not\equiv 1$ as the general solution. If $\alpha \neq 0$, substitution that $h(x) = \alpha f(x)$ in (9) gives

$$f(x + y) = f(x) + f(y). \quad (13)$$

Thus f is both additive and multiplicative. It follows that $f(x) = x$. In this case, we obtain the solution

$$F(x,y) = x, \quad H(x,y) = \alpha y + \lambda(x - 1),$$

where $\alpha \neq 0$.

■

Exercises

4.1 Determine all continuous functions $f, g, h : \mathbb{R} \to \mathbb{R}$ which satisfy the relation

$$f(x + y) = g(x) + h(y),$$

for all $x, y \in \mathbb{R}$.

4.2 Find all continuous functions $f, g, h, k : \mathbb{R} \to \mathbb{R}$ which satisfy the equation

$$f(x+y) + g(x-y) = 2h(x) + 2k(y),$$

for all reals x, y.

4.3 Find all continuous solutions of the equation

$$f(x+y) + g(x-y) = h(x)k(y),$$

where f, g, h, k are real valued functions defined on $\mathbb{R}$.

4.4 Find all continuous functions $f : \mathbb{R}^n \to \mathbb{R}$ which satisfy

$$f(x_1 + y_1, x_2 + y_2, \ldots x_n + y_n) = f(x_1, x_2, \ldots x_n) + f(y_1, y_2, \ldots y_n),$$

for all $(x_1, x_2, \ldots x_n)$ and $(y_1, y_2, \ldots y_n)$ in $\mathbb{R}^n$.

4.5 Determine all continuous functions $f : \mathbb{R} \to \mathbb{C}$ which satisfy

(a) $f(x_1 + x_2 + \cdots + x_{2002}) = f(x_1) f(x_2) \ldots f(x_{2002})$;

(b) $\overline{f(2002)} f(x) = f(2002) \overline{f(x)}$.

(Here $\overline{z}$ denotes the *complex conjugate* of z.)

4.6 Find all continuous functions $f, g, h : \mathbb{R} \to \mathbb{R}$ which satisfy the equation

$$f(x+y) - f(x-y) = 2g(x)h(y).$$

4.7 Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x+y)+f(y+z)+f(z+x) = f(x)+f(y)+f(z)+f(x+y+z),$$

for all $x, y, z \in \mathbb{R}$.

4.8 Find all continuous functions $f, g, h : \mathbb{R} \to \mathbb{R}$ such that

$$f(x+y)+f(y+z)+f(z+x) = g(x)+g(y)+g(z)+h(x+y+z),$$

for all $x, y, z \in \mathbb{R}$.

4.9 Find all continuous functions $f, g, h : \mathbb{R}^+ \to \mathbb{R}$ which are such that

$$f(x+y) + g(xy) = h(x) + h(y),$$

for all $x, y \in \mathbb{R}^+$.

4.10 Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x+y) + f(x)f(y) = f(xy+1).$$

4.11 Find all $f : \mathbb{R} \to \mathbb{R}$ which are continuous and which satisfy the equation

$$f(x+y) - f(x-y) = 2f(xy+1) - f(x)f(y) - 4,$$

for all $x, y \in \mathbb{R}$.

4.12 Find all continuous functions $f, g, h : \mathbb{R} \to \mathbb{R}$ which satisfy

$$f(x+y) + g(xy) = h(x)h(y) + 1,$$

for all $x, y \in \mathbb{R}$.

4.13 Find all continuous functions $f, g, h : \mathbb{R} \to \mathbb{R}$ which satisfy

$$f(x+y) + h(x)h(y) = g(xy+1),$$

for all $x, y \in \mathbb{R}$.

4.14 Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x+y) + f(xy-1) = f(x) + f(y) + f(xy),$$

for all reals x, y and $f(1) = 2$.

4.15 Find all continuous functions $f, g, h : \mathbb{R} \to \mathbb{R}$ which satisfy the equation

$$f(x+y) = g(x)h(y),$$

for all $x, y \in \mathbb{R}$.

4.16 Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous function such that

$$f(x_1) + f(x_2) + f(x_3) = f(y_1) + f(y_2) + f(y_3),$$

for all real numbers x_1, x_2, x_3, y_1, y_2, y_3 for which $x_1 + x_2 + x_3 = y_1 + y_2 + y_3$. Prove that f is linear; i.e., $f(x) = \alpha x + \beta$ for some real numbers α and β.

4.17 Find all continuous functions $f : \mathbb{R} \to \mathbb{C}$ such that

$$f(x)f(y) = f\left(\sqrt{x^2+y^2}\right),$$

for all $x, y \in \mathbb{R}$.

4.18 Find all functions $f : \mathbb{C} \to \mathbb{C}$ satisfying the equation

$$f(z_1 + z_2) = f(z_1) + f(z_2),$$

for all z_1, z_2 in $\mathbb{C}$.

4.19 Find all functions $f, g, h : \mathbb{R} \to \mathbb{R}$ which satisfy the equation

$$f(x+y) + g(x-y) = h(x),$$

for all reals x, y.

5

Equations with Additional Hypothesis

In the previous chapter, we have seen how even a simple looking equation like Cauchy's equation can lead to difficulties in its analysis. Without additional hypothesis on the function involved, the class of functions representing the given equation(s) may be quite difficult to comprehend. Even some additional milder conditions on the functions help us to fix the class of solutions satisfied by the given functional equation(s). We explore some such conditions which are of immense help in solving an equation.

Some a priori condition(s) on the required functions will tell us some nice, intrinsic properties of those functions and we make use of these properties to solve a given problem. For example:

(a) If $f : (a, b) \to \mathbb{R}$ is continuous, then f has intermediate value property and the range of f is an interval;

(b) If f is a continuous bijection on $\mathbb{R}$, then it is strictly monotone;

(c) If $f : \mathbb{R} \to \mathbb{R}$ is a strictly monotone surjection, then it is continuous;

(d) If f is a strictly increasing bijection on $\mathbb{R}$, then f is continuous and f^{-1}, the inverse of f, is also a strictly increasing continuous bijection;

(e) If $f : [a, b] \to \mathbb{R}$ is continuous, then f is bounded and it attains its bounds;

(f) If $f : \mathbb{R} \to \mathbb{R}$ is a bijection and has intermediate value property, then f is continuous.

The simplest property we can impose on a function f is that it should be a polynomial. Apart from constant functions and the identity function, the simplest function we can think of is a polynomial. One definitive advantage of polynomial functions is some understanding of their structure in terms of their *zeros*. We know that any given polynomial has a finitely many number of zeros, not exceeding the degree of the polynomial. We also know by the *fundamental theorem of algebra* that a non constant polynomial with complex coefficients always possesses a complex zero. This leads to a complete factorization of a polynomial with complex coefficients. We can make use of these properties while solving a functional equation in the class of polynomials.

Problem 5.1 Find all polynomials $P(x)$ such that

$$xP(x-1) = (x-15)P(x). \tag{1}$$

Solution: We observe that x divides $P(x)$. Thus we can find a polynomial $P_1(x)$ such that $P(x) = xP_1(x)$. Putting this in (1) and effecting suitable cancellations , we obtain

$$(x-1)P_1(x-1) = (x-15)P_1(x). \tag{2}$$

This shows that $(x-1)$ divides $P_1(x)$. If we introduce $P_1(x) = (x-1)P_2(x)$ and substitute this in (2), we obtain

$$(x-2)P_2(x-1) = (x-15)P_2(x).$$

We can also write this in the form

$$x(x-1)(x-2)P_2(x-1) = (x-15)P(x).$$

Continuing this process, we obtain a polynomial $P_{15}(x)$ such that

$$x(x-1)(x-2)\cdots(x-15)P_{15}(x-1) = (x-15)P(x)$$
$$= x(x-1)(x-2)\cdots(x-15)P_{15}(x).$$

Thus we arrive at the conclusion that $P_{15}(x-1) = P_{15}(x)$. If $P_{15}(x)$ is not a constant polynomial, then $P_{15}(x) = 0$ has a root α in $\mathbb{C}$. But then $P_{15}(\alpha - 1) = P_{15}(\alpha) = 0$ so that $\alpha - 1$ is also a root of $P_{15}(x) = 0$. Continuing this argument, we see that $\alpha, \alpha - 1, \alpha - 2, \ldots$ are all roots of $P_{15}(x) = 0$. This is clearly impossible since the equation $P_{15}(x) = 0$ can have at most finitely many roots. We conclude that $P_{15}(x)$ is a constant and hence

$$P(x) = cx(x-1)(x-2)(x-3)\cdots(x-14),$$

for some constant c. ■

Problem 5.2 Find all non constant polynomials $P(x)$ such that

$$P(x)P(x+1) = P(x^2+x+1). \tag{1}$$

Solution: Suppose α is a root of $P(x) = 0$. Then so is $\alpha^2 + \alpha + 1$. Changing x to $x - 1$ in (1), we see that

$$P(x-1)P(x) = P((x-1)^2 + (x-1) + 1) = P(x^2 - x + 1).$$

Since $P(\alpha) = 0$, we also see that $\alpha^2 - \alpha + 1$ is a root of $P(x) = 0$.

Choose α to be root of maximum modulus. (If there are several roots with maximum modulus, we may take any one of them.) This choice of α implies that $|\alpha^2 + \alpha + 1| \leq |\alpha|$ and $|\alpha^2 - \alpha + 1| \leq |\alpha|$, since both $\alpha^2 + \alpha + 1$ and $\alpha^2 - \alpha + 1$

are then roots of $P(x) = 0$. We observe that $\alpha \neq 0$. Thus we obtain

$$\begin{aligned} 2|\alpha| &= |\alpha^2 + \alpha + 1 - (\alpha^2 - \alpha + 1)| \\ &\leq |\alpha^2 + \alpha + 1| + |\alpha^2 - \alpha + 1| \\ &\leq |\alpha| + |\alpha| = 2|\alpha|. \end{aligned}$$

Since equality holds in the inequality, it follows that $\alpha^2 + \alpha + 1 = -\lambda(\alpha^2 - \alpha + 1)$ for some positive constant λ. If $|\alpha^2 + \alpha + 1| < |\alpha^2 - \alpha + 1|$, then we see that $|\alpha^2 - \alpha + 1| > |\alpha|$. Similarly, $|\alpha^2 - \alpha + 1| < |\alpha^2 + \alpha + 1|$ implies that $|\alpha^2 + \alpha + 1| > |\alpha|$. In either case we obtain a root of $P(x) = 0$ with larger modulus than that of α. We conclude that $|\alpha^2 + \alpha + 1| = |\alpha^2 - \alpha + 1|$. This shows that $\lambda = 1$ and hence

$$\alpha^2 + \alpha + 1 = -(\alpha^2 - \alpha + 1).$$

This forces that $\alpha^2 + 1 = 0$

[We can also infer this by noting that if $\alpha^2 + 1 \neq 0$, then $\alpha, \alpha^2 + \alpha + 1, \alpha^2 - \alpha + 1, -\alpha$ form the vertices of a parallelogram and hence either $|\alpha^2 - \alpha + 1| > |\alpha|$ or $|\alpha^2 + \alpha + 1| > |\alpha|$. But this contradicts the maximality of $|\alpha|$.]

Thus $\alpha = \pm i$ and hence $x^2 + 1$ is a factor of $P(x)$. We may now write

$$P(x) = (x^2 + 1)^m Q(x),$$

where $Q(x)$ is a polynomial not divisible by $x^2 + 1$. Putting this back in (1), we see that $Q(x)$ also satisfies

$$Q(x)Q(x + 1) = Q(x^2 + x + 1).$$

If at all $Q(x) = 0$ has a root, then the above analysis shows that the roots of maximal modulus must be $\pm i$. But this is impossible since $x^2 + 1$ does not divide $Q(x)$. We conclude that $Q(x)$ reduces to a constant, say, c. Putting this in the equation satisfied by Q, we get $c = 1$. Thus the class of

polynomials obeying the equation (1) is $P(x) = (1+x^2)^m$ for some positive integer m. ■

Problem 5.3 Find all polynomials $P(x)$ such that

$$P(x)P(x+1) = P(x^2).$$

Solution: Suppose α is a root of $P(x) = 0$. Then the given relation shows that $\alpha^2, \alpha^4, \alpha^8, \ldots$ are also roots of $P(x) = 0$. It follows that $|\alpha| = 0$ or $|\alpha| = 1$, for otherwise we get an infinite set of roots of $P(x) = 0$. Similarly $\alpha - 1$ is a root of $P(x) = 0$ and hence each of $(\alpha-1)^2, (\alpha-1)^4, \ldots$ is a root. We conclude again that $|\alpha - 1| = 0$ or 1. Suppose $|\alpha| = 1$ and $|\alpha - 1| = 1$. Writing $\alpha = \cos\theta + i\sin\theta$, we see that $2\cos\theta = 1$. Thus $\cos\theta = 1/2$, giving us $\theta = \pi/3$ or $5\pi/3$. If $\theta = \pi/3$, consider α^2 which is also a root of $P(x) = 0$. Then $\alpha^2 - 1$ is also a root of $P(x) = 0$ and

$$\left|\alpha^2 - 1\right|^2 = \left(\cos\frac{2\pi}{3} - 1\right)^2 + \sin^2\frac{2\pi}{3} = 3.$$

Thus we have a root $\alpha^2 - 1$ of $P(x) = 0$ which is of absolute value > 1. But then this leads to an infinite set of roots of $P(x) = 0$. Similarly is the case when $\theta = 5\pi/3$. We conclude that $\alpha = 0$ or $\alpha - 1 = 0$. This implies that $P(x)$ is of the form $cx^m(1-x)^n$, for some constant c and nonnegative integers m, n. Substituting this in the given equation, it is easy to check that $m = n$ and $c = 0$ or $c = (-1)^m$. Thus the class of polynomials satisfying the given relation is $P(x) = (-x)^m(1-x)^m$ where $m \geq 0$ is an integer or $P(x) \equiv 0$. ■

Problem 5.4 Determine all polynomials $P(x)$ with rational coefficients such that for all $|x| \leq 1$,

$$P\left(x\right) = P\left(\frac{-x + \sqrt{(3-3x^2)}}{2}\right). \qquad (1)$$

Solution: Suppose $P(x)$ is not constant. Putting $x = 0$ in (1), we get

$$P(0) = P\left(\frac{\sqrt{3}}{2}\right). \tag{2}$$

We observe that x divides $P(x) - P(0)$ by division algorithm. Similarly $x - \frac{\sqrt{3}}{2}$ divides $P(x) - P\left(\frac{\sqrt{3}}{2}\right)$. Hence the relation (2) shows that $x\left(x - \frac{\sqrt{3}}{2}\right)$ divides $P(x) - P(0)$. But $P(x) - P(0)$ is also a polynomial with rational coefficients. Since $\frac{\sqrt{3}}{2}$ is a root of $P(x) - P(0) = 0$ so is $-\frac{\sqrt{3}}{2}$. We conclude that $x + \frac{\sqrt{3}}{2}$ also divides $P(x) - P(0)$. It follows that $x\left(x - \frac{\sqrt{3}}{2}\right)\left(x + \frac{\sqrt{3}}{2}\right)$ divides $P(x) - P(0)$. Thus we infer that $3x - 4x^3$ is a factor of $P(x) - P(0)$. We write

$$P(x) = P(0) + (3x - 4x^3)P_1(x), \tag{3}$$

where $P_1(x)$ is a polynomial with rational coefficients. We observe that the transformation

$$x \to \frac{-x + \sqrt{3 - 3x^2}}{2}$$

fixes $3x - 4x^3$; i.e.,

$$3\left(\frac{-x + \sqrt{3 - 3x^2}}{2}\right) - 4\left(\frac{-x + \sqrt{3 - 3x^2}}{2}\right)^3 = 3x - 4x^3.$$

Thus changing x to $\frac{-x + \sqrt{3 - 3x^2}}{2}$ in (3) and using (1), we obtain

$$P_1(x) = P_1\left(\frac{-x + \sqrt{3 - 3x^2}}{2}\right).$$

We also notice that $\deg P_1(x) = \deg P(x) - 3$. Now by an easy induction, we obtain

$$P(x) = a_0 + a_1(3x - 4x^3) + a_2(3x - 4x^3)^2 + \cdots + (3x - 4x^3)^k Q(x),$$

where $Q(x)$ is polynomial with rational coefficients and deg $Q(x) \leq 2$. Moreover $Q(x)$ also satisfies (1). If $Q(x)$ is not constant, then it must be at least of degree 3, since $3x - 4x^3$ necessarily divides $Q(x) - Q(0)$. Hence we conclude that $Q(x)$ reduces to a constant, say, a_k. Thus the general form of the polynomial with rational coefficients obeying (1) is

$$P(x) = \sum_{j=0}^{k} a_j (3x - 4x^3)^j,$$

where a_j are rational numbers.

We remark that this argument is also valid if we consider real polynomials. ■

Problem 5.5 Find all real polynomials $P(x)$ having only real zeroes and which satisfy the equation

$$P(x)P(-x) = P(x^2 - 1). \tag{1}$$

Solution: If α is a zero of $P(x)$, then $\alpha^2 - 1$ is also a zero of $P(x)$. Thus $\alpha, \alpha^2 - 1, (\alpha^2 - 1)^2 - 1, \ldots$ are all roots of $P(x) = 0$. Hence this sequence must be periodic.

Suppose $\alpha = \alpha^2 - 1$. Then α satisfies the equation $x^2 - x - 1 = 0$, and hence $\alpha = \phi$ or $\alpha = \bar{\phi}$, where

$$\phi = \frac{1 + \sqrt{5}}{2}, \quad \bar{\phi} = \frac{1 - \sqrt{5}}{2}.$$

In this case $x - \phi$ or $x - \bar{\phi}$ is a factor of $P(x)$.

The second possibility is $(\alpha^2-1)^2-1=\alpha$. In this case we can check that $\alpha=0,-1,\phi$ or $\bar{\phi}$. But we notice that $0,1$ always go together; i.e., $\alpha=0$ is a root of $P(x)=0$ if and only if $\alpha=-1$ is a root of $P(x)=0$. This follows from $P(-1)=P(0)^2$ and $P(0)=P(1)P(-1)$. Thus $x(x+1)$ is also a factor of $P(x)$ in this case.

Let us set

$$p_{j,k,l}(x)=(x^2+x)^j(x-\phi)^k(x-\bar{\phi})^l.$$

We notice that $p_{j,k,l}(x)$ is a solution of (1). Thus by dividing $P(x)$ from $p_{j,k,l}(x)$ for some suitable j,k,l, we obtain a polynomial $Q(x)$ satisfying

$$Q(x)Q(-x)=Q(x^2-1), \tag{2}$$

which has real coefficients, which has only real zeroes but with an additional advantage that none of $x^2+x, x-\phi, x-\bar{\phi}$ is a factor of $Q(x)$. We show $Q(x)$ must reduce to a constant.

Suppose $Q(x)$ is not a constant so that it has real zeroes. Let α_0 be the least root of $Q(x)=0$. If $\alpha_0>\phi$, then $\alpha_0^2-1>\alpha_0$ and the sequence $\alpha_0, \alpha_0^2-1, (\alpha_0^2-1)^2-1,\cdots$ is strictly increasing. Hence $\alpha_0<\phi$. If $\bar{\phi}<\alpha_0<\phi$, then $\alpha_0^2-1<\alpha_0$, contradicting the choice of α_0. We conclude that $\alpha_0<\bar{\phi}<\phi$.

Since α_0 is a root of $Q(x)=0$, it follows that $x-\alpha_0$ divides $Q(x)$. Hence $x^2-1-\alpha_0$ divides $Q(x^2-1)=Q(x)Q(-x)$. Since $Q(x)$ has only real zeroes, $Q(x^2-1)$ factors completely in to linear factors. This implies, in particular, that $x^2-1-\alpha_0$ is the product of two linear factors and hence $1+\alpha_0>0$. Thus we get

$$-1<\alpha_0<\bar{\phi}<\phi.$$

We now write

$$\begin{aligned}(\alpha_0^2-1)^2-1 &= \alpha_0+\left\{(\alpha_0^2-1)^2-1-\alpha_0\right\}\\ &= \alpha_0+\alpha_0(\alpha_0+1)(\alpha_0-\phi)(\alpha_0-\bar{\phi}).\end{aligned}$$

Since $-1 < \alpha_0 < \bar{\phi} < 0 < \phi$, we see that the product on the right hand side is negative. Hence we conclude that

$$(\alpha_0^2 - 1)^2 - 1 < \alpha_0.$$

But $(\alpha_0^2-1)^2-1$ is also a root of $Q(x) = 0$. This contradicts the choice of α_0. It follows that $Q(x)$ is a constant. It is easy to check that this constant is 0 or 1.

Thus the solutions to our problem are polynomials of the form

$$P(x) = (x^2 + x)^j (x - \phi)^k (x - \bar{\phi})^l,$$

where j, k, l are nonnegative integers or $P(x) \equiv 0$. ■

Problem 5.6 Find all polynomial $P(x, y)$ in two variables which satisfy

$$P(x, y) = P(x + 1, y + 1). \tag{1}$$

Solution: Let us put

$$u = \frac{x+y}{2}, \quad v = \frac{x-y}{2}, \quad P(x, y) = f(u, v).$$

We see that

$$u + 1 = \frac{x+y}{2} + 1 = \frac{(x+1)+(y+1)}{2},$$
$$v = \frac{x-y}{2} = \frac{(x+1)-(y+1)}{2}.$$

This shows that

$$f(u + 1, v) = P(x + 1, y + 1) = P(x, y) = f(u, v),$$

where we have used (1). It follows by an easy induction that $f(u+n, v) = f(u, v)$ for all natural numbers. We also observe that $f(u - 1, v) = f(u, v)$ and hence $f(u + n, v) = f(u, v)$ for all integers n.

Fix $(a,b) \in \mathbb{R}^2$ and consider the polynomial

$$g(u) = f(u,b) - f(a,b).$$

We observe that $g(a) = 0$ and $g(u+n) = g(u)$ for all $n \in \mathbb{Z}$. Thus g is a constant polynomial and this constant is 0 since $g(a) = 0$. We conclude that $f(u,b) = f(a,b)$ for all $u \in \mathbb{R}$. Taking $a = 0$ and $b = v$, we get $f(u,v) = f(0,v)$, for all $u, v \in \mathbb{R}$. It follows

$$P\Big(x,y\Big) = P\left(0, \frac{x-y}{2}\right).$$

This shows that $P(x,y)$ is a polynomial in $x-y$; i.e.,

$$P(x,y) = \sum a_j (x-y)^j.$$

■

If we think that the imposition of polynomial behavior on the function is stringent, then we can look for milder conditions: monotonicity, Darboux property(also called as intermediate value property), continuity, boundedness or differentiability. Each of them has its own advantage, but may not solve the functional equation in its generality.

Problem 5.7 Let $n \geq 2$ be a fixed integer. Determine all bounded functions $f : (0,a) \to \mathbb{R}$ which satisfy

$$f(x) = \frac{1}{n^2}\left\{ f\left(\frac{x}{n}\right) + f\left(\frac{x+a}{n}\right) + \cdots + f\left(\frac{x+(n-1)a}{n}\right)\right\}.$$

Solution: Suppose $\big|f(x)\big| \leq M$ for all $x \in (0,a)$. Putting this in the given relation, we obtain

$$\Big|f(x)\Big| \leq \frac{1}{n^2}\Big(M + M + \cdots M\Big),$$

where there are n summands. We conclude that

$$\left|f(x)\right| \leq \frac{M}{n},$$

for all $x \in (0, a)$. Thus we have reduced the bound on $|f|$ from M to M/n. We can make use of this fresh bound to get a still smaller bound:

$$\left|f(x)\right| \leq \frac{M}{n^2}, \quad \text{for all } x \in (0, a).$$

We continue by induction to prove that

$$\left|f(x)\right| \leq \frac{M}{n^k}, \quad \text{for all } x \in (0, a),$$

and for every natural number k. It follows that $f(x) = 0$ for all $x \in (0, a)$. ■

Problem 5.8 Find all strictly monotone functions $f : \mathbb{R} \to \mathbb{R}$ satisfying the functional equation

$$f\big(f(x) + y\big) = f\big(x + y\big) + f\big(0\big),$$

for all x, y in $\mathbb{R}$.

Solution: Taking $y = -x$, we see that $f\big(f(x) - x\big) = 2f\big(0\big)$. Since f is strictly monotone, it is one-one. It follows that $f(x) - x$ is constant. Thus we get $f(x) = x + c$, for some constant c. It is easy to verify that such an f is strictly increasing and satisfies the functional equation. ■

Problem 5.9 Find all monotonic functions $f : \mathbb{R} \to \mathbb{R}$ which satisfy the functional equation

$$f\big(x + f(y)\big) = f\big(x\big) + y^n, \quad \text{for all } x, y \in \mathbb{R}, \tag{1}$$

where n is a fixed positive integer.

(Romanian Competitions-1998)

Solution: Suppose y_1 and y_2 are two positive real numbers such that $f(y_1) = f(y_2)$. Then the equation (1) shows that $y_1^n = y_2^n$. This forces that $y_1 = y_2$, because a positive real number possesses a unique positive n-th root. Thus f is an one-one function on $(0, \infty)$.

Choose $x_0 > 0$ such that $x_0 + f(0) > 0$. Then we see from (1) that

$$f\big(x_0 + f(0)\big) = f\big(x_0\big).$$

Since f is one-one on the set of all positive real numbers, we conclude that $x_0 + f(0) = x_0$. Thus it follows that $f(0) = 0$ and hence (1) reduces to

$$f\big(f(y)\big) = y^n, \quad \text{for all } y \in \mathbb{R}. \tag{2}$$

We also observe that

$$f\Big(f\big(x + f(y)\big)\Big) = f\big(f(x) + y^n\big) = f\big(y^n\big) + x^n,$$

for all $x, y \in \mathbb{R}$.Invoking (2), we obtain

$$\begin{aligned}\big(x + f(y)\big)^n &= f\Big(f\big(x + f(y)\big)\Big) = f\big(y^n\big) + x^n \\ &= f\Big(f\big(f(y)\big)\Big) + x^n = f\big(y\big)^n + x^n,\end{aligned}$$

valid for all real x, y. Taking $x = 1$ and $y = f(1)$, we see that

$$2^n = 2.$$

This forces $n = 1$ and hence $f\big(f(y)\big) = y$ for all real numbers y. It follows that f is also onto and one-one on the whole of $\mathbb{R}$. Taking $y = f(z)$ in (1), we get

$$f(x + z) = f(x) + f(z), \quad \text{for all } x, z \in \mathbb{R}.$$

Thus f is a monotonic function which satisfy Cauchy's equation. We conclude that $f(x) = cx$ for some constant c. Using $f\big(f(y)\big) = y$ for all y, it follows that $c^2 = 1$. This gives $c = \pm 1$. Thus we obtain two solutions $f(x) = x$ for all $x \in \mathbb{R}$ and $f(x) = -x$ for all $x \in \mathbb{R}$ in the case $n = 1$ and there is no solution for $n > 1$. ∎

Problem 5.10 Suppose $f : \mathbb{R}^+ \to \mathbb{R}^+$ is a strictly decreasing function which satisfy the relation

$$f(x+y)+f\Big(f(x)+f(y)\Big) = f\Big(f\big(x+f(y)\big)+f\big(y+f(x)\big)\Big), \tag{1}$$

for all $x, y \in \mathbb{R}^+$. Prove that $f\big(f(x)\big) = x$ for all $x \in \mathbb{R}^+$.

(Iranian Competitions-1999)

Solution: Putting $y = x$ in (1), we obtain

$$f(2x) + f\big(2f(x)\big) = f\big(2f\big(x + f(x)\big)\big). \tag{2}$$

Changing x to $f(x)$ in (2), we also get

$$f\big(2f(x)\big) + f\big(2f\big(f(x)\big)\big) = f\Big(2f\Big(f(x) + f\big(f(x)\big)\Big)\Big). \tag{3}$$

Subtracting (2) from (3), we obtain

$$\begin{aligned} &f\Big(2f\big(f(x)\big)\Big) - f(2x) \\ &= f\Big(2f\Big(f(x) + f\big(f(x)\big)\Big)\Big) - f\Big(2f\big(x + f(x)\big)\Big). \end{aligned} \tag{4}$$

Suppose $f\big(f(x)\big) > x$, for some x. Then $f\Big(2f\big(f(x)\big)\Big) < f(2x)$ because f is strictly decreasing. Thus the left hand side of (4) is negative, forcing the inequality $f\Big(2f\Big(f(x) + f\big(f(x)\big)\Big)\Big) < f\Big(2f\big(x + f(x)\big)\Big)$. Invoking the strictly decreasing nature of f twice, we observe that $f(x)+f\big(f(x)\big) < x + f(x)$. This implies that $f\big(f(x)\big) < x$ contradicting what we have started with. Similarly, we can show that $f\big(f(x)\big) < x$ is not possible. We conclude that $f\big(f(x)\big) = x$ for all $x \in \mathbb{R}^+$. ∎

Problem 5.11 Find all functions $h : \mathbb{R} \to \mathbb{R}$ for which there exists strictly monotone function $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x+y) = f(x)h(y) + f(y), \tag{1}$$

for all reals x, y.

(Romanian Competitions-1999)

Solution: We show that $h(x) = a^x$ for some positive real a. In fact, if $a = 1$, then we can take $f(x) = x$ so that (1) is true. If $a \neq 1$, then we can choose $f(x) = a^x - 1$, and it is easy to verify (1) in this case.

Suppose such a function f exists. Taking $y = 0$ in (1), we get $f(0) = \big(1 - h(0)\big)f(x)$, valid for all reals x. If $h(0) \neq 1$, then we can solve for $f(x)$ and we see that $f(x)$ is a constant function. But then it cannot be strictly monotone. It follows that $h(0) = 1$ and $f(0) = 0$. Since f is strictly monotone, we can further infer that $f(x) \neq 0$ if $x \neq 0$. Using the symmetry of the left hand side of (1) in x and y, we obtain

$$f(x)h(y) + f(y) = f(y)h(x) + f(x).$$

If x, y are such that $xy \neq 0$, then we can solve this to get

$$\frac{h(x) - 1}{f(x)} = \frac{h(y) - 1}{f(y)}.$$

Thus $\dfrac{h(x) - 1}{f(x)} = K$, for some constant K, and this is true for all $x \neq 0$. We can therefore write $h(x) = 1 + Kf(x)$, for all $x \neq 0$. Since $h(0) = 1$ and $f(0) = 0$, this is also valid for $x = 0$.

If $K = 0$, then $h(x) = 1$ for all x. Suppose $K \neq 0$. Then we see that

$$\begin{aligned} h(x+y) &= 1 + Kf(x+y) \\ &= 1 + Kf(x)h(y) + Kf(y) = h(x)h(y). \end{aligned}$$

By an easy induction, we get $h(nx) = (h(x))^n$ for all x and $n \in \mathbb{N}$. We also observe that $h(x)h(-x) = h(0) = 1$, so that $h(-x) = 1/h(x)$ for all reals x. This implies that $h(nx) = (h(x))^n$ for all integers n. Since f is strictly monotone and $h(x) = 1 + Kf(x)$, we also see that $h(x)$ is strictly monotone. This implies that $h(x) > 0$ for all reals x. Let us take $a = h(1)$. Then $a > 0$ and $h(n) = (h(1))^n = a^n$ for all integers n. If $r = p/q$ is a rational, then

$$h\left(\frac{p}{q}\right)^q = h(p) = a^p,$$

so that $h(r) = a^r$ for all rationals r. Since h is also strictly monotone, we conclude that $h(x) = a^x$ for all reals x. ■

Problem 5.12 Find all continuous functions $f : \mathbb{R} \to \mathbb{R}_0$ such that

$$f(x+y) = f(x) + f(y) + f(x)f(y), \tag{1}$$

for all reals x, y.

Solution: Adding 1 to both sides of (1), we obtain

$$\begin{aligned} 1 + f(x+y) &= 1 + f(x) + f(y) + f(x)f(y) \\ &= \big(1 + f(x)\big)\big(1 + f(y)\big). \end{aligned}$$

If we set $g(x) = 1 + f(x)$, then we see that

$$g(x+y) = g(x)g(y).$$

Since $g(x) \geq 1$, for all real x, the function $\ln g(x)$ is defined for all real x. Moreover the continuity of f also implies that of g and hence that of $h(x) = \ln g(x)$. We also infer that

$$h(x+y) = h(x) + h(y), \quad \text{for all } x, y \in \mathbb{R}.$$

Thus h is a continuous additive function on $\mathbb{R}$. We have proved earlier (see chapter 4) that such a function is necessarily of the form $h(x) = cx$ where $c = h(1)$. Going back we see that $f(x) = \exp(cx) - 1$. ■

Problem 5.13 Find all continuous functions f from reals to it self such that

$$\big(1+f(x)f(y)\big)f\big(x+y\big)=f\big(x\big)+f\big(y\big),$$

for all real numbers x, y.

(Samasya, Vol.8, No.1, May, 2001)

Solution: Consider the equation

$$\big(1+f(x)f(y)\big)f\big(x+y\big)=f\big(x\big)+f\big(y\big). \tag{1}$$

Taking $x=y=0$ in (1), we obtain $\big(f(0)\big)^3=f\big(0\big)$. Thus $f(0)$ lies in the set $\{-1,0,1\}$. We consider these possibilities separately.

Case 1. Suppose $f(0)=-1$. Taking $y=0$ in (1), we obtain

$$\big(1-f(x)\big)f\big(x\big)=f\big(x\big)-1.$$

This shows that either $f(x)=1$ or $f(x)=-1$ for each x. The continuity of f shows that $f(x)$ is either identically equal to 1 or identically equal to -1. Since $f(0)=-1$, we conclude that $f(x)=-1$ for all real x.

Case 2. If $f(0)=1$, then again (1) gives

$$\big(1+f(x)\big)f\big(x\big)=f\big(x\big)+1,$$

so that $f(x)=1$ or $f(x)=-1$. Using the continuity of f and $f(0)=1$, we conclude that $f(x)=1$ for all real x.

Case 3. Finally suppose $f(0)=0$. The relation (1) shows that f is an odd function; i.e., $f(-x)=-f(x)$ for all $x\in\mathbb{R}$. Replacing x and y in (1) by $x/2$, we obtain

$$f\big(x\big)=\frac{2f\big(x/2\big)}{1+f\big(x/2\big)^2}, \tag{2}$$

for all real x. We know that for any real number t, the inequality

$$\left|\frac{2t}{1+t^2}\right| \leq 1,$$

holds. Thus (2) shows that $|f(x)| \leq 1$ for all real x.

Suppose $f(x_0) = 1$ for some $x_0 \in \mathbb{R}$. Using (2), we obtain $f(x_0/2) = 1$. By an easy induction, we prove that

$$f\left(\frac{x_0}{2^n}\right) = 1,$$

for all natural numbers. By continuity of f, we obtain

$$1 = \lim_{n\to\infty} f\left(\frac{x_0}{2^n}\right) = f(0).$$

This contradicts $f(0) = 0$. We conclude that there is no x_0 such that $f(x_0) = 1$. Similarly we can rule out $f(x_0) = -1$. It follows that $|f(x)| < 1$ for all real x.

Since $|f(x)| < 1$ for all real x, we can find a continuous function $g : \mathbb{R} \to \mathbb{R}$ such that $f(x) = \tanh g(x)$. Using (1), we obtain

$$\tanh g(x+y) = \frac{\tanh g(x) + \tanh g(y)}{1 + \tanh g(x) \tanh g(y)} = \tanh\big(g(x)+g(y)\big).$$

Since $t \to \tanh(t)$ is an one-one function on $\mathbb{R}$, we conclude that

$$g(x+y) = g(x) + g(y),$$

for all reals x, y. Now the continuity of g shows that $g(x) = cx$ for some real constant c. Thus we get $f(x) = \tanh cx$ for all real x.

The above analysis shows that there are three functions satisfying (1): (i) $f_1(x) = 1$ for all $x \in \mathbb{R}$; (ii) $f_2(x) = -1$ for all $x \in \mathbb{R}$; and (iii) $f_3(x) = \tanh cx$ for all $x \in \mathbb{R}$, where c is a real constant. ∎

Problem 5.14 Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ which are solutions of the functional equation

$$f(x + yf(x)) = f(x)f(y), \text{ for all } x, y \in \mathbb{R}. \tag{1}$$

Solution: First we show that any continuous non constant function which satisfy (1) assumes a nonzero real number at most once. Suppose for some $x_1 < x_2$, we have $f(x_1) = f(x_2) \neq 0$. Taking $x = x_1$ and $y = (t - x_1)/f(x_1)$ in (1), we obtain

$$f(t) = f\left(x_1 + \frac{t - x_1}{f(x_1)} f(x_1)\right) = f(x_1) f\left(\frac{t - x_1}{f(x_1)}\right).$$

Taking $x = x_2$ and $y = (t - x_1)/f(x_1)$ in (1), we also obtain

$$f\left(x_2 + \frac{t - x_1}{f(x_1)} f(x_2)\right) = f(x_2) f\left(\frac{t - x_1}{f(x_1)}\right).$$

Using $f(x_1) = f(x_2)$ and comparing these expressions, we see that

$$\begin{aligned} f(t) &= f(x_1) f\left(\frac{t - x_1}{f(x_1)}\right) \\ &= f(x_2) f\left(\frac{t - x_1}{f(x_1)}\right) \\ &= f\left(x_2 + \frac{t - x_1}{f(x_1)} f(x_2)\right) \\ &= f\left(x_2 + \frac{t - x_1}{f(x_1)} f(x_1)\right) = f(t + x_2 - x_1). \end{aligned}$$

This shows that f is periodic with period $x_2 - x_1$. Since f is continuous and periodic, it is bounded. Let x_0 be point

such that $|f(x_0)| = \sup|f(x)|$. If $f(x_0) = 0$, then we get $f(x) = 0$ for all real x. Otherwise, in each neighborhood of x_0, we can find $a < b$ such that $f(a) = f(b) \neq 0$. The above analysis shows that f is periodic with period $b - a$. Thus f has arbitrarily small periods. Since f is also continuous, it follows that f is constant, contrary to our assumption. Thus if f is a non constant continuous function which obey (1), then it can take a nonzero value at most once.

If f is constant, the only possibilities are $f(x) \equiv 0$ and $f(x) \equiv 1$. Suppose f is a continuous non constant solution of (1). Suppose $x \neq y$ and $xyf(x)f(y) \neq 0$. We have

$$f\big(x + yf(x)\big) = f(x)f(y) = f\big(y + xf(y)\big) \neq 0.$$

But f assumes any nonzero real number at most once. We conclude that $x + yf(x) = y + xf(y)$ and hence

$$\frac{f(x) - 1}{x} = \frac{f(y) - 1}{y}.$$

Thus if $x \neq 0$ and $f(x) \neq 0$, then $\big(f(x) - 1\big)/x$ is constant. We see that $f(0) = 1$ unless $f(x) \equiv 0$. We conclude that every continuous non constant solution satisfy $f(0) = 1$ and for every $x \neq 0$ either $f(x) = 0$ or $f(x) = 1 + cx$ for some constant c. Hence the non constant continuous solutions are

$$\begin{aligned}
f(x) &= 1 + cx; \\
f(x) &= \begin{cases} 1 - \dfrac{x}{\alpha} & \text{if } x \leq \alpha\,, \\ 0 & \text{for } x \geq \alpha > 0; \end{cases} \\
f(x) &= \begin{cases} 1 - \dfrac{x}{\alpha} & \text{for } x \geq \alpha, \\ 0 & \text{for } x \leq \alpha < 0. \end{cases}
\end{aligned}$$

Remarks: If we don't insist on continuous solutions, then we can construct other solutions. For example, the Dirichlet's function defined by $f(x) = 0$ for all rationals and

$f(x) = 1$ for all irrationals is a solution of (1). This is an example of a non continuous, but bounded (even *measurable*) function which satisfy (1). Using the idea of Hamel basis, we can construct even unbounded (and *non measurable*) non continuous solutions of (1).

∎

Problem 5.15 Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ satisfying the relation

$$f\big(f(x)\big) = f\big(x\big) + 2x, \quad \text{for all } x \in \mathbb{R}. \tag{1}$$

(Belarusian Mathematical Olympiad-1998)

Solution: First of all we observe that f is an one-one function. If $f(x) = f(y)$ for some reals x, y, then $f\big(f(x)\big) = f\big(f(y)\big)$ and (1) shows that $x = y$.

Since f is continuous and one-one, it is a strictly monotonic function. Otherwise f would assume some value twice contradicting the one-one nature of f. Now (1) also implies that $f\big(f(0)\big) = f\big(0\big)$ and using the fact that f is an one-one function, we can conclude that $f(0) = 0$. We distinguish two possibilities: either f is strictly increasing or f is strictly decreasing.

Case 1. Suppose f is is strictly increasing on $\mathbb{R}$. Since $f(0) = 0$, it follows that $f(x) > 0$ if and only if $x > 0$. Since f is strictly increasing and continuous, it has a continuous inverse, say, g; i.e., $f\big(g(x)\big) = x$ for all x in the range of f and $g\big(f(x)\big) = x$ for all real x. Let us put

$$h(x) = f(x) - 2x, \quad x \in \mathbb{R}.$$

We observe that $h(0) = 0$ and

$$\begin{aligned} h\big(f(x)\big) &= f\big(f(x)\big) - 2f\big(x\big) \\ &= f(x) + 2x - 2f(x) \\ &= 2x - f(x) = -h(x). \end{aligned}$$

Replacing x by $f(x)$ in this relation, we obtain

$$h\big(f^2(x)\big) = -h\big(f(x)\big) = h(x).$$

By an easy induction, it follows that

$$h\big(f^{2n}(x)\big) = h(x), \quad \text{for all } x \in \mathbb{R}, \quad n \in \mathbb{N}.$$

(Here f^k is the k-fold composition of f.) It is easy to check that this is valid for all integers n.

Since $f(x) > 0$ if and only if $x > 0$, we observe that for each x, the numbers $x, f(x)$ and $f\big(f(x)\big)$ all have the same sign and

$$2f^{-2}(x) = x - f^{-1}(x).$$

This shows, in view of same sign of $x, f^{-1}(x)$ and $f^{-2}(x)$, that

$$\big|f^{-2}(x)\big| \leq \frac{|x|}{2}.$$

Using induction on n, we obtain

$$\big|f^{-2n}(x)\big| \leq \frac{|x|}{2^n},$$

for all $n \in \mathbb{N}$. It follows now that $f^{-2n}(x) \to 0$ as $n \to \infty$, for each x. Since f is continuous, so is h. Thus, we get

$$\begin{aligned} h(x) &= h\Big(f^{-2n}(x)\Big) \\ &= \lim_{n\to\infty} h\Big(f^{-2n}(x)\Big) \\ &= h\Big(\lim_{n\to\infty} f^{-2n}(x)\Big) \\ &= h(0) = 0. \end{aligned}$$

It follows that $f(x) = 2x$ for all $x \in \mathbb{R}$.

Case 2. Suppose f is strictly decreasing on $\mathbb{R}$. Since $f(0) = 0$, it follows that $f(x) > 0$ if $x < 0$ and $f(x) < 0$ if $x > 0$.

Thus we see that $f(f(x))$ is strictly increasing. We put $h(x) = f(x) + x$. We observe that

$$h(f(x)) = f(f(x)) + 2f(x) = 2f(x) + 2x = 2h(x).$$

By induction, we obtain

$$h(x) = \frac{h(f^{2n}(x))}{2^{2n}},$$

for all $n \in \mathbb{N}$. We note that x and $f(f(x))$ have the same sign and it is different from that of $f(x)$. Hence (1) gives

$$|f^2(x)| \leq 2|x|, \quad \text{for all } x \in \mathbb{R}.$$

Again by induction, it follows that

$$|f^{2n}(x)| \leq 2^n|x|.$$

Since $f^2(x) = x + h(x)$, we conclude that $|h(x)| \leq |x|$. Thus we obtain

$$\begin{aligned} |h(x)| &= \left|\frac{h(f^{2n}(x))}{2^{2n}}\right| \\ &\leq \frac{|f^{2n}(x)|}{2^{2n}} \\ &\leq \frac{2^n|x|}{2^{2n}} = \frac{|x|}{2^n}. \end{aligned}$$

This shows that $h(x) = 0$ for all $x \in \mathbb{R}$. We conclude that $f(x) = -x$.

It is easy to verify that these two functions $f(x) = 2x$ and $f(x) = -x$ are indeed solutions of our problem. ∎

Problem 5.16 Find all strictly increasing bijective functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x) + f^{-1}(x) = 2x,$$

for all real x. (Here $f^{-1}(x)$ denotes the inverse of f.)

(Asian Pacific Mathematical Olympiad-1989)

Solution: Let f be a solution of the given functional equation. For each real number α, let us put

$$S_\alpha = \{x \in \mathbb{R} : f(x) - x = \alpha\}.$$

We show that S_α is nonempty for exactly one value of α. Since f is strictly increasing and a bijection, it is continuous. Hence the range of $f(x) - x$ is an interval, possibly degenerate(i.e., reduced to a point). For any α in the range of $f(x) - x$, we see that S_α is a nonempty set. If we show that S_α is not empty for at most one value of α, we obtain the desired result. We prove this in several steps.

(I). If $x \in S_\alpha$, then $x + \alpha \in S_\alpha$. Indeed, $x \in S_\alpha$ implies that $f(x) - x = \alpha$ and hence $f^{-1}(x + \alpha) = f^{-1}(f(x)) = x$. But the given equation shows that $f(x+\alpha) = 2(x+\alpha) - f^{-1}(x+\alpha)$. We thus obtain $f(x + \alpha) = 2(x + \alpha) - x = x + 2\alpha$. This shows that $f(x+\alpha) - (x+\alpha) = \alpha$, implying that $x+\alpha \in S_\alpha$.

(II). If $x \in S_\alpha$ and $y \geq x$, then $y \notin S_\beta$ for any $\beta < \alpha$. Suppose $x \leq y < x + (\alpha - \beta)$. Then by the monotonicity of f, we obtain $f(y) \geq f(x) = x + \alpha$. If $y \in S_\beta$, then $y + \beta = f(y) \geq x + \alpha$, proving $y \geq x + (\alpha - \beta)$. We thus conclude that $y \notin S_\beta$, in this case.

Now we show that if $x+(n-1)(\alpha-\beta) \leq y < x+n(\alpha-\beta)$, then $y \notin S_\beta$. We have proved this for $n = 1$. Suppose this is true for all $k < n$ and y is such that $x + (n-1)(\alpha - \beta) \leq y < x + n(\alpha - \beta)$. Then we see that

$$x + \alpha + (n-2)(\alpha - \beta) \leq y + \beta < x + \alpha + (n-1)(\alpha - \beta).$$

But we know from (I) that $x \in S_\alpha$ implies that $x + \alpha \in S_\alpha$. By induction hypothesis, it follows that $y + \beta \notin S_\beta$. Again by (I), we conclude that $y \notin S_\beta$.

Since given any y, we can sand-witch it between $x+n(\alpha-\beta)$ and $x + (n+1)(\alpha - \beta)$, for some n, the result follows by induction.

(III). If for some $\alpha < \beta$, we have S_α is nonempty and S_β is nonempty, then S_γ is also nonempty for each γ such that $\alpha < \gamma < \beta$. Since S_α not an empty set, we can find $a \in S_\alpha$ such that $f(a) - a = \alpha$. Similarly, we can find $b \in S_\beta$ such that $f(b) - b = \beta$. Since $f(x) - x$ is continuous, it has intermediate value property. Thus we can find c such that $f(c) - c = \gamma$, and this proves that S_γ is nonempty.

We use these three properties to prove that S_α is a nonempty set for at most one value of α. Suppose $\beta < \alpha$ are such that S_α and S_β are nonempty sets. If $\beta > 0$, then (I) shows that for each $y \in S_\beta$, the real number $y + k\beta \in S_\beta$ for every natural number k. If $x \in S_\alpha$, choose k such that $y + k\beta \geq x$. But then $y + k\beta \notin S_\beta$ by (II). If $\beta < \alpha < 0$, then we can find arbitrarily large negative numbers in S_α, from (I). Hence for any given $y \in S_\beta$, we can find $x \in S_\alpha$ such that $x < y$. But then y cannot be in S_β, by (II). If $\beta < 0 \leq \alpha$, then S_0 is not empty from (III). Choose γ such that $\beta < \gamma < 0$. Again by (III), S_γ is not empty. But this reduces to the second case. Similarly, we can dispose off $\beta \leq 0 < \alpha$. We conclude that S_α is a nonempty set for at most one value of α.

We have seen in the beginning that for each α in the range of $f(x) - x$, the set S_α is nonempty. Thus there exist a unique α such that $f(x) - x = \alpha$ for all real numbers x. We conclude that the class of solutions of the given equation is $f(x) = x + d$, where d is a real number.

Alternate Solution: Consider the relation $f(x) + f^{-1}(x) = 2x$. Replacing x by $f(x)$ we obtain $f^2(x) = 2f(x) - x$. [Here we use $f^n(x) = f(f^{n-1}(x))$.] Replacing x again by $f(x)$, we see that

$$f^3(x) = 2f^2(x) - f(x) = 2(2f(x) - x) - f(x) = 3f(x) - 2x.$$

By an easy induction we get

$$f^n(x) = nf(x) - (n-1)x,$$

for all $n \in \mathbb{N}$ and $x \in \mathbb{R}$. Since f is strictly increasing, it follows that $f^n(x) > f^n(0)$ if $x > 0$ and $f^n(x) < f^n(0)$ if $x < 0$. But we can compute that

$$\begin{aligned} f^n(x) - f^n(0) &= nf(x) - (n-1)x - nf(0) \\ &= n\big(f(x) - x - f(0)\big) + x, \end{aligned}$$

so that

$$\frac{f^n(x) - f^n(0)}{n} - \frac{x}{n} = f(x) - x - f(0).$$

Letting $n \to \infty$ and using the fact that $f^n(x) > f^n(0)$ if $x > 0$ and $f^n(x) < f^n(0)$ if $x < 0$, we arrive at the conclusion $f(x) - x - f(0) \geq 0$ for $x > 0$ and $f(x) - x - f(0) \leq 0$ for $x < 0$.

Now replacing x by $f^{-1}(x)$ in the given relation, we also obtain $f^{-2}(x) = 2f^{-1}(x) - x$ and using $f^{-1}(x) = 2x - f(x)$, we get $f^{-2}(x) = 3x - 2f(x)$. Using induction, we can prove that

$$f^{-n}(x) = (n+1)x - nf(x),$$

valid for all natural numbers n and real numbers x. Since f is strictly increasing, so is f^{-1} and we conclude that $f^{-n}(x) > f^{-n}(0)$ if and only if $x > 0$. But $f^{-n}(x) - f^{-n}(0) = x - n\big(f(x) - x - f(0)\big)$ and this shows that

$$\frac{f^{-n}(x) - f^{-n}(0)}{n} - \frac{x}{n} = -\big(f(x) - x - f(0)\big).$$

Letting $n \to \infty$, we conclude that $f(x) - x - f(0) \leq 0$ for $x > 0$ and $f(x) - x - f(0) \geq 0$ for $x < 0$

Combining all these observations, we obtain $f(x) = x + f(0)$ for all x. Thus the class of functions which satisfy the given functional equation is $f(x) = x + d$, where d is a real constant. ∎

Problem 5.17 Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ which satisfy the relation

$$3f(2x+1) = f(x) + 5x,$$

for all real x.

Solution: It is easy to see that $f(x) = x - \frac{3}{2}$ is a solution. We show that this is the only solution of our problem. Put $g(x) = f(x) - \left(x - \frac{3}{2}\right)$. Then g is also continuous on $\mathbb{R}$. Fix a positive number N. Since g is continuous on $[-N, N]$, it is bounded there. Let $|g(x)| \leq M$ for all x in $[-N, N]$. It is easy to check that g satisfies the relation

$$3g(2x+1) = g(x).$$

Thus for $x \in [-N, N]$, we obtain

$$\left|g(2x+1)\right| \leq \frac{M}{3}.$$

Note that $\{2x+1 \ : \ x \in [-N, N]\}$ contains the interval $[-N, N]$. We conclude that $|g(x)| \leq M/3$ for all $x \in [-N, N]$. Thus we have reduced the bound from M to $M/3$. Using this fresh bound and the relation satisfied by g, we can reduce the bound to $M/3^2$. Now the argument is clear. We can show by induction that

$$\left|g(x)\right| \leq \frac{M}{3^n},$$

for every natural number n and for all $x \in [-N, N]$. It follows that $g(x) = 0$ for all $x \in [-N, N]$. Since N is at our disposal, we conclude that $g(x) = 0$ for all real x. ∎

Some problems do not need the full force of continuity. It may be sufficient to use the fact that a continuous function has intermediate value property (i.e., the range of a real valued continuous function on $\mathbb{R}$ is always an interval). Such a function is often called a *Darboux* function or we say that the function has *Darboux* property.

Problem 5.18 Find all functions $f : \mathbb{R} \to \mathbb{R}$ with Darboux property such that for some $n \geq 1$,

$$f^n(x) = -x, \quad \text{for all } x \in \mathbb{R}.$$

(Here f^n denotes the n-fold composition of f with it self.)

Solution: We first observe that f is a bijection on $\mathbb{R}$. Indeed for any x, we have $x = f\big(f^{n-1}(-x)\big)$ so that it is on to. If $f(x) = f(y)$ for some reals x, y, then we obtain $f^n(x) = f^n(y)$ so that $x = y$. Thus f is also one-one. Since f has intermediate value property, this in conjunction with bijectivity implies that f is a monotonic function. Moreover, we see that

$$f(-x) = f^{n+1}(x) = f^n\big(f(x)\big) = -f(x),$$

showing that f is an *odd* function. It follows that $f(0) = 0$. We claim that f is a decreasing function. Suppose the contrary. Since we know that f is monotone, it must be an increasing function. Thus $x < y$ implies that $f(x) < f(y)$. Taking composition with f, we obtain $f^n(x) < f^n(y)$. Thus we finally get $-x < -y$ or $y < x$. This contradiction shows that f is decreasing. Thus x and $f(x)$ have different signs. We conclude that $xf(x) < 0$ for all $x \neq 0$.

Let $x_0 > 0$ and define $x_k = f\big(x_{k-1}\big)$ for $k \geq 1$. Then it is easy to see that $(-1)^k x_k > 0$ and $x_n = f^n\big(x_0\big) = -x_0$. Thus we obtain

$$(-1)^{n+1} x_0 = (-1)^n x_n > 0.$$

Since $x_0 > 0$, we conclude that n is odd. Suppose $x_1 > -x_0$. Since f is decreasing and an *odd* function, we obtain

$$x_2 = f\big(x_1\big) < f\big(-x_0\big) = -f\big(x_0\big) = -x_1.$$

This shows that

$$(-1)^2 x_2 < (-1)^1 x_1.$$

By induction we can prove that

$$(-1)^k x_k > (-1)^{k+1} x_{k+1}.$$

Thus we have

$$x_0 > (-x_1) > x_2 > \cdots > (-x_n) = x_0.$$

This contradiction shows that $x_1 > -x_0$ is not possible. Similarly we can show that $x_1 < -x_0$ also leads to a contradiction. We conclude that $x_1 = -x_0$. This leads to $f(x_0) = -x_0$ for all $x_0 > 0$. Using the fact that f is *odd*, we conclude that $f(x) = -x$ for all $x \in \mathbb{R}$. ∎

Problem 5.19 Let α be a fixed nonzero real number. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$. such that

$$f\left(2x - \frac{f(x)}{\alpha}\right) = \alpha x, \quad \text{for all } x \in \mathbb{R}. \tag{1}$$

(AMM-1984)

Solution: We show that $f(x) = \alpha(x-c)$ for some constant c. Let us put

$$g(x) = 2x - \frac{f(x)}{\alpha}, \quad x \in \mathbb{R}.$$

Then g is continuous on $\mathbb{R}$. Moreover,

$$g(g(x)) = 2g(x) - \frac{f(g(x))}{\alpha} = 2g(x) - x.$$

This property of g shows that g is one-one. Since g is continuous, it follows that g is strictly monotone. We also see that

$$g(g(x)) - g(x) = g(x) - x. \tag{2}$$

We claim that g is strictly increasing on $\mathbb{R}$. Suppose this is not the case. The monotonicity of g forces that it is a

decreasing function. Hence if $x < y$, then $g(x) > g(y)$ and hence $g\big(g(x)\big) < g\big(g(y)\big)$. Thus we obtain

$$g\big(g(x)\big) - g(x) < g\big(g(y)\big) - g(y).$$

This gives $g(x) - x < g(y) - y$, in view of (2). But we also know that $x < y$ implies $g(x) > g(y)$ and hence $g(x) - x > g(y) - y$. Thus we arrive at two contradictory statements. We conclude that g is indeed a strictly increasing function.

We also observe that

$$\begin{aligned} g^2(x) &= 2g(x) - x, \\ g^3(x) = g^2\big(g(x)\big) &= 2g^2(x) - g(x) \\ &= 2\big(2g(x) - x\big) - g(x) = 3g(x) - 2x. \end{aligned}$$

By an easy induction, we prove that

$$g^n(x) = ng(x) - (n-1)x, \quad \text{for all } x \in \mathbb{R}, n \geq 1.$$

Thus we get

$$\begin{aligned} g^n(x) - g^n(0) &= ng(x) - (n-1)x - ng(0) \\ &= n\big(g(x) - x - g(0)\big) + x. \end{aligned} \tag{3}$$

Since g is increasing, we see that $g^n(x) - g^n(0) > 0$ if $x > 0$ and $g^n(x) - g^n(0) < 0$ if $x < 0$. Using (3), we obtain

$$\frac{g^n(x) - g^n(0)}{n} - \frac{x}{n} = g(x) - x - g(0).$$

The above expression shows that

$$\lim_{n\to\infty} \frac{g^n(x) - g^n(0)}{n} = g(x) - x - g(0),$$

for all x. However for $x > 0$, we have

$$\lim_{n\to\infty} \frac{g^n(x) - g^n(0)}{n} \geq 0,$$

and for $x < 0$, we have

$$\lim_{n\to\infty} \frac{g^n(x) - g^n(0)}{n} \leq 0.$$

These imply that

$$g(x) \geq g(0) + x, \quad \text{for } x > 0,$$

$$g(x) \leq g(0) + x, \quad \text{for } x < 0.$$

The above conditions also imply that g is on to. Let $s = \inf\{g(x) : x \in \mathbb{R}\}$ and $r = \sup\{g(x) : x \in \mathbb{R}\}$. Suppose $r < \infty$. Taking $x = r + 1 - g(0)$, we see that x is a positive real and hence

$$g\big(r + 1 - g(0)\big) \geq g(0) + r + 1 - g(0) \geq r + 1.$$

But this contradicts the definition of r. Thus $r = \infty$. Similarly we can prove that $s = -\infty$. This shows that the range of g is $(-\infty, \infty)$ and hence g is on to.

Since g is one-one and on to, g^{-1} exists and hence $g^n(x)$ is defined for all $n \in \mathbb{Z}$ and

$$\frac{g^n(x) - g^n(0)}{n} - \frac{x}{n} = g(x) - x - g(0), \quad \text{for all } n \in \mathbb{Z}.$$

Letting $n \to -\infty$, we see that $g(x) - x - g(0) \leq 0$ for all $x > 0$. Similarly $g(x) - x - g(0) \geq 0$ for $x < 0$. It follows that $g(x) - x - g(0) = 0$ for all x. We thus conclude that $f(x) = \alpha(x - c)$, where $c = g(0)$. ∎

Problem 5.20 Suppose $f : \mathbb{R} \to \mathbb{R}$ is a twice continuously differentiable function such that

$$2f(x + 1) = f(x) + f(2x), \forall x \in \mathbb{R}. \tag{1}$$

Prove that f is a constant function.

Solution: Replacing x by $x-k$ in (1), we get

$$2f(x-k+1) = f(x-k) + f\big(2(x-k)\big). \tag{2}$$

Since f is twice differentiable, we get from (2) after successively differentiating the expression

$$2f'(x-k+1) = f'(x-k) + 2f'\big(2(x-k)\big); \tag{3}$$

$$2f''(x-k+1) = f''(x-k) + 4f''\big(2(x-k)\big). \tag{4}$$

Putting $x = k+1$ in (3), we obtain $f'(1) = 0$. We observe that for all large k, the intervals $[-k, k]$ and $[-k+1, k+1]$ are contained in $[-2k, 2k]$. Hence for all $x \in [0, 2k]$, we see that $x-k+1, x-k$ and $2(x-k)$ all lie in $[-2k, 2k]$. Since f is continuously twice differentiable, f'' is continuous. Hence it is bounded on $[-2k, 2k]$. Suppose $|f''(y)| \leq M$ for all $y \in [-2k, 2k]$. Then (4) shows that

$$4\big|f''\big(2(x-k)\big)\big| \leq \big|f''(x-k)\big| + 2\big|f''(x-k+1)\big|.$$

As x varies from 0 to $2k$, we see that $2(x-k)$ varies from $-2k$ to $2k$. Hence using the bound for $|f''|$, we obtain

$$4\big|f''(y)\big| \leq 3M, \quad \text{for all } y \in [-2k, 2k].$$

This gives us a fresh bound for $|f''|$ on $[-2k, 2k]$:

$$\big|f''(y)\big| \leq \frac{3}{4}M, \quad \text{for all } y \in [-2k, 2k].$$

Using this fresh bound we deduce again from (4) that

$$\big|f''(y)\big| \leq \left(\frac{3}{4}\right)^2 M, \quad \text{for all } y \in [-2k, 2k].$$

By iterating this process, we obtain

$$\big|f''(y)\big| \leq \left(\frac{3}{4}\right)^n M, \quad \text{for all } y \in [-2k, 2k],$$

for every natural number n. Since $(3/4)^n \to 0$ as $n \to \infty$, we conclude that $f''(y) = 0$ for all $y \in [-2k, 2k]$. Thus $f'(y)$ is constant on $[-2k, 2k]$. Since $f'(1) = 0$, it follows that $f'(y) = 0$ for all $y \in [-2k, 2k]$. This in turn implies that $f(y)$ is constant on $[-2k, 2k]$. But this is true for every large k. We conclude that f is a constant function on $\mathbb{R}$. ∎

In the following few problems, differentiability is a consequence of the given relation and it is exploited to solve the problem.

Problem 5.21 Suppose $f : \mathbb{R} \to \mathbb{C}$ satisfies the equation

$$f(x+y) = f(y) + e^{i\alpha y} f(x), \quad \text{for all } x, y \in \mathbb{R}, \tag{1}$$

where $\alpha \neq 0$ is a real constant. Show that

$$f(y) = c\,\frac{(e^{i\alpha y} - 1)}{i\alpha}, \quad \text{for all } y \in \mathbb{R},$$

where c is some constant.

Solution: We observe that $f(0) = 0$. Interchanging x and y in (1), we also obtain

$$f(y+x) = f(x) + e^{i\alpha x} f(y), \ x, y \in \mathbb{R}. \tag{2}$$

Comparing (1) and (2), we see that

$$f(y)\left(e^{i\alpha x} - 1\right) = f(x)\left(e^{i\alpha y} - 1\right).$$

Taking $x = \pi/\alpha$ in this expression, we see that

$$f(y) = -\frac{f\left(\frac{\pi}{\alpha}\right)}{2}\left(e^{iky} - 1\right).$$

This expression shows that f is a differentiable function. We can write (1) also in the form

$$\frac{f(x+y) - f(y)}{x} = e^{i\alpha y}\frac{f(x)}{x}.$$

Using the differentiability of f and letting $x \to 0$, we get

$$f'(y) = f'(0)e^{i\alpha y}.$$

Solving this differential equation, we obtain

$$f(y) = f'(0)\frac{e^{i\alpha y}}{i\alpha} + A,$$

for some constant A. Using $f(0) = 0$, we conclude that

$$A = -\frac{f'(0)}{i\alpha}.$$

This leads to

$$f(y) = c\,\frac{(e^{i\alpha y} - 1)}{i\alpha}, \quad \text{for all } y \in \mathbb{R},$$

where $c = f'(0)$. ■

Note in the last problem that if $\alpha = 0$, then f satisfies Cauchy's equation and we need continuity on f to get a nice description of it.

Problem 5.22 Find all $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that f^{-1} exists on $\mathbb{R}^+$ and

$$xy \leq \frac{1}{2}\left\{xf(x) + yf^{-1}(y)\right\}, \text{ for all } x, y \in \mathbb{R}^+, \qquad (1)$$

where $\mathbb{R}^+$ denotes the set of all positive real numbers.

Solution: Since f^{-1} exists on $\mathbb{R}^+$, it follows that f is both one-one and on to on $\mathbb{R}^+$. If $x, y \in \mathbb{R}^+$ and $z = f(y)$, then we get

$$2xf(y) = 2xz \leq xf(x) + zf^{-1}(z) = xf(x) + yf(y).$$

Interchanging x and y, we obtain

$$2yf(x) \leq yf(y) + xf(x).$$

Adding these two expressions, we obtain

$$xf(y) + yf(x) \leq xf(x) + yf(y).$$

This leads to

$$(x - y)f(y) \leq (x - y)f(x).$$

If $x > y$, then we get $f(x) \geq f(y)$. If $x < y$, then we get $-f(y) \leq -f(x)$ or $f(x) \leq f(y)$. Thus $f(x)$ is a nondecreasing function of x. We also obtain

$$\frac{y - x}{y} f(x) \leq f(y) - f(x) \leq \frac{y - x}{x} f(y). \tag{2}$$

Since f is nondecreasing and on to, it is continuous on $\mathbb{R}^+$. Now (2) shows that

$$\frac{f(x)}{y} \leq \frac{f(y) - f(x)}{y - x} \leq \frac{f(y)}{x}.$$

for $y > x$ and

$$\frac{f(y)}{x} \leq \frac{f(y) - f(x)}{y - x} \leq \frac{f(x)}{y},$$

for $y < x$. The continuity of f shows that

$$\lim_{y \to x} \frac{f(y) - f(x)}{y - x} = \frac{f(x)}{x}.$$

Thus f is differentiable on $\mathbb{R}^+$ and

$$f'(x) = \frac{f(x)}{x}.$$

Solving this differential equation, we obtain $f(x) = cx$ for some constant $c > 0$. ∎

Problem 5.23 Let α, β be given real numbers. Find all differentiable functions $f : \mathbb{R} \to \mathbb{R}$ which obey the equation

$$f'(\alpha x + \beta y) = \frac{f(y) - f(x)}{y - x}, \quad \text{for all } x, y \in \mathbb{R}, x \neq y. \quad (1)$$

Solution: We show that $f(x) = ax + b$, for some real constants, except when $\alpha = \beta = 1/2$ in which case $f(x)$ could be any quadratic polynomial with real coefficients.

Suppose $\alpha + \beta \neq 0$. Taking

$$x = \frac{z - \beta}{\alpha + \beta}, \quad y = \frac{z + \alpha}{\alpha + \beta},$$

in (1), we obtain

$$f'(z) = f(y) - f(x).$$

This shows that f' itself is differentiable. We may assume $\beta \neq 0$ and write (1) in the form

$$(y - x) f'(\alpha x + \beta y) = f(y) - f(x).$$

Differentiating this with respect to x, we obtain

$$-f'(\alpha x + \beta y) + \alpha(y - x) f''(\alpha x + \beta y) = -f'(x).$$

If we take $y = -\frac{\alpha}{\beta} x$, the above expression reduces to

$$f'(x) = f'(0) + \left(\alpha + \beta\right)\left(\frac{\alpha}{\beta}\right) f''(0) x.$$

It follows that f is a polynomial of degree at most 2. Let us put

$$f(x) = ax^2 + bx + c,$$

where a, b, c are real numbers. Then we see that

$$f'(x) = 2ax + b.$$

Substituting this in (1), we obtain

$$2a(\alpha x + \beta y) = a(x + y) + b.$$

The above expression shows that $a \neq 0$ if and only if $2(\alpha x + \beta y) = x + y$. Taking $x = 0$ and $y \neq 0$, we see that $\beta = 1/2$. Similarly taking $x \neq 0$ and $y = 0$, we conclude that $\alpha = 1/2$. Thus f is a linear polynomial except when $\alpha = \beta = 1/2$ in which case f is a quadratic polynomial.

If $\alpha + \beta = 0$, let us put $y = x + t$, where $t \neq 0$. Then (1) takes the form

$$f'(\beta t) = \frac{f(x + t) - f(x)}{t}, \quad \text{for all } x \in \mathbb{R}.$$

Differentiating with respect to x, we obtain

$$f'(x + t) - f'(x) = 0.$$

Since this is true for all real x, we can substitute $x = 0$ to get $f'(t) = f'(0)$. It follows that $f'(t)$ is a constant function on $\mathbb{R}$. We conclude that f is a linear polynomial; i.e., a polynomial of degree 1. ∎

Exercises

5.1 Find all polynomials $P(x)$ which satisfy the relation

$$P(x + 1) = P(x) + 2x + 1.$$

5.2 What are the continuous functions on $\mathbb{R}$ which are solutions of the equation

$$xf(y) + yf(x) = (x + y)f(x)f(y)?$$

5.3 Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

1. f is continuous at 0;
2. $f(x+y) = f(x) + f(y) + xy(x+y)$ for all reals x, y.

5.4 Find all polynomials $P(x)$ which satisfy the equation

$$P(x)P\big(2x^2\big) = P\big(2x^3 + x\big).$$

5.5 Find all polynomials $P(x)$ with complex coefficients such that

$$P(x)P(-x) = P\big(x^2\big).$$

5.6 Find all polynomials $P(x)$ such that

$$P\big((x+1)^2\big) = P\big(x^2\big) + 2x + 1.$$

5.7 Find all polynomials $P(x)$ which are solutions of the equation

$$P\big(x^2 - y^2\big) = P(x+y)P(x-y).$$

5.8 A rational function f (i.e., a function which is the quotient of two polynomials) has the property that $f(x) = f\big(1/x\big)$. Prove that f is a function in the variable $x + \dfrac{1}{x}$.

5.9 Let $\mathbb{R}^+$ denote the set of all positive real numbers. Find all $f : \mathbb{R}^+ \to \mathbb{R}^+$ which satisfy

1. $f\big(xf(y)\big) = yf(x)$ for all $x, y \in \mathbb{R}^+$;
2. $f(x) \to 0$ as $x \to \infty$.

(IMO-1983)

5.10 Let f, g be two bijective continuous functions on $\mathbb{R}$ such that

$$f\big(g^{-1}(x)\big) + g\big(f^{-1}(x)\big) = 2x,$$

for all $x \in \mathbb{R}$. Suppose there exists $x_0 \in \mathbb{R}$ such that $f(x_0) = g(x_0)$. Prove that $f(x) = g(x)$ for all real numbers x.

5.11 Find all polynomials $P(x)$ for which

$$P(2x) = P'(x)P''(x)$$

holds. (Here $P'(x)$ and $P''(x)$ are respectively the first and second derivative of P.)

5.12 Find all continuous functions $f : (a, b) \to \mathbb{R}$ satisfying

$$f(xyz) = f(x) + f(y) + f(z),$$

for all $x, y, z \in (a, b)$ under the restriction $1 < a^3 < b$.

(Contests from Higher Mathematics)

5.13 Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ obeying the relation

$$f(xy) = xf(y) + yf(x),$$

for all $x, y \in \mathbb{R}$.

5.14 Find all polynomials $P(x)$ with real coefficients such that

$$xP(x - n) = (x - 1)P(x)$$

for some $n \in \mathbb{N}$ and for all $x \in \mathbb{R}$.

(Croatian National Mathematics Competition-1994)

5.15 Suppose f, g are two continuous functions on $\mathbb{R}$ such that

$$f(x - y) = f(x)f(y) + g(x)g(y),$$

for all reals x, y. Prove that

$$g(x - y) = g(x)f(y) - g(y)f(x),$$

for all $x, y \in \mathbb{R}$, without solving the equation explicitly.

5.16 Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x+y)f(x-y) = \big(f(x)f(y)\big)^2,$$

for all real numbers x, y.

5.17 Find all polynomials $P(x)$ satisfying the equation

$$(x-16)P(2x) = 16(x-1)P(x),$$

for all x. (Irish Mathematical Olympiad-1997)

5.18 Let $n \geq 3$ be an arbitrary integer. Find all continuous functions $f : [0,1] \to \mathbb{R}$ such that

$$f(x_1) + f(x_2) + f(x_3) + \cdots + f(x_n) = 1,$$

for all $x_1, x_2, x_3, \ldots, x_n$ in $[0,1]$ which satisfy $x_1 + x_2 + x_3 + \cdots + x_n = 1$. (Crux-1992)

5.19 Find all differentiable functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x+y) = f(x-y) + y\big[f'(x+y) + f'(x-y)\big],$$

for all $x, y \in \mathbb{R}$.

5.20 Let $a > 0$ be a real number. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ which satisfy the equation

$$f(x+y) = a^{xy} f(x) f(y),$$

for all real numbers x, y.

5.21 Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ which satisfy the equation

$$f(x+y) = \frac{f(x) + f(y)}{1 - f(x)f(y)},$$

for all x, y.

5.22 Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ which satisfy the equation

$$f(x+y) = \frac{f(x) + f(y) + 2f(x)f(y)}{1 - f(x)f(y)},$$

for all x, y.

5.23 Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ which satisfy the equation

$$f(x+y+axy) = f(x)f(y),$$

for all x, y.

5.24 Find all continuous functions $f : \mathbb{R} \to \mathbb{R}_0$ which satisfy the equation

$$f\left(\frac{x+y}{2}\right) = \sqrt{f(x)f(y)},$$

for all x, y.

5.25 Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ which satisfy the equation

$$f\left(\frac{x+y}{1+(xy/K^2)}\right) = f(x)f(y),$$

for all x, y.

6

Additional Problems

Problem 6. 1 (IMO-2002)Find all functions from the set $\mathbb{R}$ of real numbers to itself such that

$$\big(f(x)+f(z)\big)\big(f(y)+f(t)\big)=f(xy-zt)+f(xt+yz).$$

Problem 6. 2 (IMOTC-2003)Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that

- $f(x+y)+f(x)f(y)=f(x)+f(y)+f(xy),$

for all reals x, y.

Problem 6. 3 (Bulgaria, 1999)Find all polynomials

$$P(x)=x^n+a_{n-1}x^{n-1}+a_{n-2}x^{n-2}+\cdots+a_1x+a_0,$$

where $a_0\neq 0$, a_j's are integers and $P(x)=0$ has roots $a_0, a_1, a_2, \ldots, a_{n-1}$.

Problem 6. 4 Find all functions $f:\mathbb{R}^+\to\mathbb{R}^+$ satisfying the equation

$$f\big(xf(y)\big)=f\big(xy\big)+x,$$

for all positive real numbers x, y.

Problem 6. 5 (Romania, 1990)Find all all polynomials $P(x)$ such that

$$2P\big(2x^2-1\big)=P\big(x\big)^2-2,$$

for all $x\in\mathbb{R}$

Problem 6. 6 (Bulgaria, 1998)Show that there is no function $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$f(x)^2 \geq f(x+y)\big(f(x)+y\big),$$

for all x, y in $\mathbb{R}^+$.

Problem 6. 7 Let $\mathbf{U}$ be the unit disc in the complex plane: $\mathbf{U} = \{z \in \mathbb{C} \big| |z| = 1\}$. Suppose $f : \mathbf{U} \to \mathbf{U}$ is a continuous function such that

$$f(z^2) = f(z)^2$$

for all $z \in \mathbf{U}$. Prove that $f(z) = z^n$ for some integer n.

Problem 6. 8 (17-th Balkan Maths Olympiad)Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f\big(xf(x) + f(y)\big) = f(x)^2 + y,$$

for all reals x, y.

Problem 6. 9 (Canada, 2002)Find all functions $f : \mathbb{N}_0 \to \mathbb{N}_0$ satisfying the equation

$$xf(y) + yf(x) = (x+y)f\big(x^2+y^2\big),$$

for all x, y in $\mathbb{N}_0$.

Problem 6. 10 Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f\big(x^2 + f(y)\big) = xf(x) + y,$$

for all real numbers x, y.

Problem 6. 11 Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying the equation

$$f\big(xf(x) + y\big) = x^2 + f(y),$$

for all $x, y \in \mathbb{R}$.

Problem 6. 12 Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying the equation

$$f\big(x^2 + yf(z)\big) = xf(x) + zf(y),$$

for all $x, y, z \in \mathbb{R}$.

Problem 6. 13 Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying the equation

$$f\big(x^2 + yf(y)\big) = xf(x) + y^2,$$

for all $x, y \in \mathbb{R}$.

Problem 6. 14 (IMOTC-2008)Find all functions $f : (0, \infty) \to (0, \infty)$ such that

$$f\big(f(x) + y\big) = xf\big(1 + xy\big),$$

for all x, y in $(0, \infty)$.

Problem 6. 15 (Macedonia, 2004)Does there exist a function $f : \mathbb{N} \to \mathbb{N}$ such that

$$f\big(f(n+1)\big) = f(n+1) - f(n),$$

for all $n \geq 2$.

Problem 6. 16 (HongKong, 2004)Find all functions $f : \mathbb{R} \to \mathbb{R}$ which satisfy the equation

$$f(x + yf(x)) = f(x) + xf(y),$$

for all real x, y.

Problem 6. 17 (South Africa, 2003.)Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x^2) - f(y^2) = (x + y)(f(x) - f(y)),$$

for all real x, y.

Problem 6. 18 (Proposed for IMO-2002)Find all functions $f : \mathbb{R} \to \mathbb{R}$ which satisfy

$$f\big(f(x) + y\big) = 2x + f\big(f(y) - x\big),$$

for all $x, y \in \mathbb{R}$.

Problem 6. 19 (IMO-2004)Find all polynomials $P(x)$ with real coefficients which satisfy the relation

$$P(a-b) + P(b-c) + P(c-a) = 2P(a+b+c),$$

for all triples a, b, c of real numbers such that $ab+bc+ca = 0$.

Problem 6. 20 (Romania, 2007)Find all $f : \mathbb{N} \to \mathbb{N}$ such that $m^2 + f(n)$ divides $f(m)^2 + n$ for all $m, n \in \mathbb{N}$.

Problem 6. 21 (Balkan Maths Olympiads, 2007)Find all $f : \mathbb{R} \to \mathbb{R}$ which satisfy the equation

$$f\big(f(x) + y\big) = f\big(f(x) - y\big) + 4f(x)y,$$

for all real numbers x, y.

Problem 6. 22 Find all polynomials $P(x)$ with real coefficients such that

$$P(x^2) = P(x)P(x+2).$$

Problem 6. 23 (Iran, 2007)Find all real polynomials P(x) of degree 3 such that $P(x+y) \geq P(x) + P(y)$ for all non-negative real numbers x, y.

Problem 6. 24 (Czeck-Slovak, 2004)Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$x^2\big(f(x) + f(y)\big) = (x+y)f\big(f(x)y\big),$$

for all $x, y \in \mathbb{R}$.

Problem 6. 25 (Sweden, 2003)Find all real polynomials $P(x)$ such that

$$1 + P(x) = \frac{1}{2}\{P(x-1) + P(x+1)\},$$

for all $x \in \mathbb{R}$.

Problem 6. 26 Find all $f : \mathbb{N} \to \mathbb{N}$ such that $f(m)^2 + f(n)$ divides $(m^2 + n)^2$ for all natural numbers m, n.

Problem 6. 27 Find all $f : \mathbb{N} \to \mathbb{N}$ which satisfy the relation

$$f\big(m - n + f(n)\big) = f(m) + f(n),$$

for all $m, n \in \mathbb{N}$.

Problem 6. 28 Show that there is no function $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$(x + y)f\big(f(x)y\big) = x^2 f\big(f(x) + f(y)\big),$$

for all positive real numbers x, y.

Problem 6. 29 Find all $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x)^2 + 2yf(x) + f(y) = f(y + f(x)),$$

for all real x, y.

Problem 6. 30 (Japan, 2007) Find all functions $f : \mathbb{R}^+ \to \mathbb{R}$ such that

$$f(x) + f(y) \leq \frac{f(x+y)}{2}, \quad \frac{f(x)}{x} + \frac{f(y)}{y} \geq \frac{f(x+y)}{x+y},$$

for all $x, y \in \mathbb{R}^+$.

Problem 6. 31 Find all functions $f : \mathbb{R}^2 \to \mathbb{R}$ such that

$$f(\lambda x, \lambda y) = \lambda^k f(x, y),$$

for all $(x, y) \in \mathbb{R}^2$.

Problem 6. 32 Find all functions $f : \mathbb{R}^2 \to \mathbb{R}$ such that

$$f(x + t, y + t) = f(x, y) + t, \quad f(xt, yt) = f(x, y)t,$$

for all $x, y, t \in \mathbb{R}$.

Problem 6. 33 Find all continuous functions $f, g, h : \mathbb{R} \to \mathbb{R}$ which satisfy the equation

$$f(x + y) = f(x)g(y) + h(y),$$

for all $x, y \in \mathbb{R}$.

Problem 6. 34 Find all non-constant polynomials $P(x)$ with real coefficients such that

$$P(x)P(x - 1) = P(x^2).$$

Problem 6. 35 [7] Find all continuous function $f : [0, 1] \to \mathbb{R}$ such that for all $x \in (0, 1)$ there exists $h > 0$ such that $0 \leq x - h < x + h \leq 1$ and

$$f(x - h) + f(x + h) = 2f(x).$$

Problem 6. 36 [7] Suppose $f : \mathbb{R} \to \mathbb{R}$ is a one-one continuous map such that

(i) $f\big(2x - f(x)\big) = x$ for all $x \in \mathbb{R}$;

(ii) There Exists $x_0 \in \mathbb{R}$ such that $f(x_0) = x_0$.

Prove that $f(x) = x$ for all real x.

Problem 6. 37 [7] Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ which satisfy the equation

$$f(x+y)f(x-y) = \big(f(x)f(y)\big)^2,$$

for all $x, y \in \mathbb{R}$.

Problem 6. 38 (Iran, 2008)Find all $f : \mathbb{R} \to \mathbb{R}$ such that

$$f\big(xf(y)\big) + y + f(x) = f\big(x + f(y)\big) + yf(x),$$

for all $x, y \in \mathbb{R}$.

Problem 6. 39 (Iran, 2008) Let k be a given natural number. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that $f(m) + f(n)$ divides $(m+n)^k$ for all natural numbers m, n.

Problem 6. 40 Suppose $f : \mathbb{R} \to \mathbb{R}$ is such that $f(x+y) \leq f(x) + f(y)$ for all real x, y and $f(x) \leq x$ for all x. Prove that $f(x) = x$ for all x. Show that the assumption $f(x) \geq x$ may not lead to the same conclusion.

Problem 6. 41 Suppose $f : \mathbb{R} \to \mathbb{R}$ satisfies $f\big(f(x)\big) = x$ and $f(\lambda x) = \lambda f(x)$ for all real numbers x and λ. Prove that $f(x) = x$ for all x or $f(x) = -x$ for all x. What if $f(\lambda x) = \lambda^k f(x)$?

Problem 6. 42 Find all functions $f : \mathbb{N} \to \mathbb{N}$ which satisfy the equation

$$f(m^2 + mn) = f(m)^2 + f(m)f(n),$$

for all natural numbers m, n.

Problem 6. 43 (UK, 2003) Let $f : \mathbb{N}_0 \to \mathbb{N}_0$ be a function such that

(i) $\big(f(2n+1)\big)^2 - \big(f(2n)\big)^2 = f(n) + 1;$

(ii) $f(2n) \geq f(n)$, for all $n \in \mathbb{N}_0$.

(Here $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.) Find the number of elements in the range of f which are ≤ 2003.

Problem 6. 44 (Romania, 2003) Find all $f : \mathbb{R} \to \mathbb{R}$ which satisfy

(i) $f(x) = x^2$, for all $x \in [0, 1]$;

(ii) $f(x + 1) = f(x) + 1$, for all $x \in \mathbb{R}$.

Problem 6. 45 Find all functions $f, g : \mathbb{R} \to \mathbb{R}$ which satisfy the equation

$$(x - y)f(z) + (y - z)f(x) + (z - x)f(y) = g(x + y + z),$$

for all real numbers x, y, z, such that $x \neq y$, $y \neq z$ and $z \neq x$.

7

Hints to Exercises

2 Equations on Natural Numbers

2.1 Show that $f(0) = 1$, $f(1) = 0$, $f(2) = -1$, $f(3) = -2$, $f(-1) = 2$, $f(-2) = 3$, etc. Can you guess the answer? Prove that your guess is correct using induction.

2.2 Suppose $f(1) = q^2$. Prove by induction that $f(n) = n(q^2 + n - 1)$ for all n. Thus for each prime p, we see that p divides $q^2 - 1$. It follows that $q = 1$ and this implies that $f(n) = n^2$.

2.3 We show that in fact there is no function $\mathbb{N}_0 \to \mathbb{N}_0$ such that $f\big(f(n)\big) = f(n)+k$, where k is any odd positive integer. Suppose such an f exists. Then we see that $f(n + k) = f(n)+k$ for all $n \in \mathbb{N}_0$. It follows that $f(n-k) = f(n)-k$ for all $n \geq k$. Consider $g(n) = f(n)-k$. Verify that $g\big(f(n)\big) = n$ for all $n \in \mathbb{N}$ and $f\big(g(n)\big) = n$ for all n such that $f(n) \geq k$. Consider the sets $A = \big\{n \mid 0 \leq n \leq k-1, f(n) < k\big\}$ and $B = \big\{n \mid 0 \leq n \leq k-1, f(n) \geq k\big\}$. Show that there is a bijection between A and B. Conclude that $2|A| = k$, where $|X|$ denotes the number of elements in a finite set X. This forces a contradiction.

2.4 Let us put $f(1) = k$. From the given equation, we get $f\big(m^2 + k\big) = 1 + f(m)^2$ and $1 + k^2 = f(1 + k)$. Use this to prove that

$$f(f(m))^2 - f(k)^2 = m^2 - 1.$$

Taking $m = 2$, conclude that $f(k) = 1$. Obtain $f\big(f(m)\big) = m$ for all $m \in \mathbb{N}$. Use this to prove that $f(n+1) = f(n)+k^2$. By induction $f(n+1) = f(1) + nk^2 = k + nk^2$. Conclude that $k = 1$ and hence $f(n) = n$ for all $n \in \mathbb{N}$.

2.5 It is easy to see that $f(2) = f(1) + 1$ and $f(4) = f(1) + 3$. Conclude that $f(3) = f(1) + 2$. By induction, get $f(n) = f(1) + n - 1$ for all $n \in \mathbf{N}$. Suppose $f(1) = m > 1$. The the numbers $(m+1)! + 2, (m+1)! + 3, \ldots, (m+1)! + (m+1)$ are all composite. Take the least prime p exceeding $(m+1)! + (m+1)$. Set $n = p - m + 1$. Then $p = f(n)$ and hence n is a prime. But $n > (m+1)! + 2$ and hence $p > n > (m+1)! + (m+1)$. This contradicts the minimality of p. Conclude that $f(1) = 1$ and hence $f(n) = n$ for all $n \in \mathbb{N}$. The answer is $f(2001) = 2001$.

2.6 Get $f\big(g(0)\big) = f(0)^2 + g(0)$ and $g\big(f(0)\big) = g(0)^2 + f(0)$. From this and the commutativity of f and g prove that $2g(0)^2 = g\big(g(0)\big)^2$. Conclude that $g(0) = 0$. Similarly prove that $f(0) = 0$. It follows that $f\big(g(n)\big) = g(n)$ and $g\big(f(n)\big) = f(n)$. This implies that $f(n) = g(n)$ for all n. Use $f(0) = 0$ to conclude that $f\big(m^2\big) = f(m)^2$. Since $f(1) > 0$, it follows that $f(1) = 1$. By induction prove that $f(n) = n$ for all n. Hence $g(n) = n$ for all n.

2.7 Let $\alpha = \dfrac{1+\sqrt{5}}{2}$ be the *golden ratio*. Let $g(n) = \alpha n$ and $f(n) = \left[g(n) + \dfrac{1}{2}\right]$, the closest integer to $g(n)$. Check that $f(1) = 2$ and $f(n+1) \geq f(n)+1 > f(n)$. Using the fact that $|f(n) - g(n)| < 1/2$ and $g\big(g(n)\big) = g(n) + n$, show that $|f\big(f(n)\big) - f(n) - n| < \dfrac{1}{2}$. Conclude that $f\big(f(n)\big) = f(n)+n$, for all natural numbers n.

2.8 Note that f is one-one . Show that $f(1) > 1$ is not possible and hence $f(1) = 1$. Suppose $f(k) = k$

for all $k < n$. If $f(n) < n$, then $f\big(f(n)\big) = f(n)$ and hence $f(n) = n$ since f is one-one. If $f(n) > n$, then $f\big(f(n)\big) \geq n$; for $f\big(f(n)\big) < n$ with induction hypothesis implies that $f\Big(f\big(f(n)\big)\Big) = f\big(f(n)\big)$ and the injectivity of f gives $f\big(f(n)\big) = f(n)$ and hence $f(n) = n$. Similarly $f\Big(f\big(f(n)\big)\Big) \geq n$. But then $f\Big(f\big(f(n)\big)\Big) + f\big(f(n)\big) + f(n) > 3n$ contradicting the given relation. Thus $f(n) = n$ and induction proves the result.

2.9 Prove that $f(0) = 0$ and hence $f(n)\{f(n)+f(-n)\} = 0$ for all $n \in \mathbb{Z}$. Changing n to $-n$, conclude that $f(n) + f(-n) = 0$. Taking $m = 2$ and $n = 1$, conclude that $\big(f(3) - 1\big)\big(f(2) - 1\big) = 2$. This gives four possibilities: (a) $f(2) = 2$, $f(3) = 3$; (b) $f(2) = 3$, $f(3) = 2$; (c) $f(2) = 0$, $f(3) = -1$; and (d) $f(2) = -1$, $f(3) = 0$. Show that (a) leads to $f(n) = n$ for all $n \in \mathbb{Z}$. Rule out (b) by proving $f(4) = 9$ and hence $8f(5) = 20$. In the case (c), obtain the function f such that $f(n) = 0$ for $n = 2k$, $f(n) = 1$ for $n = 4k+1$ and $f(n) = -1$ for $n = 4k+3$. Similarly (d) leads to the function f given by $f(n) = 0$ for $n = 3k$, $f(n) = 1$ for $n = 3k+1$ and $f(n) = -1$ for $n = 3k + 2$.

2.10 Consider $F(n) = [\alpha n]$ where α denotes the golden ratio; $\alpha = \dfrac{1+\sqrt{5}}{2}$. Prove that $F(1) = 1$ and $0 < F(n+1) - F(n) < 3$. Thus $F(n+1) - F(n) = 1$ or 2. Show that $H(n) = F(F(n)-n+1)$ satisfies $n-1 < H(n) < n+2$ so that $H(n) = n$ or $n+1$. Prove that $F(n+1) - F(n) = 1$ leads to $H(n) > n$ and hence $H(n) = n+1$. If $F(n+1) - F(n) = 2$, then prove that $H(n) < n + 1$ to conclude that $H(n) = n$. Thus F and f satisfy the same recurrence relation and same initial conditions.

2.11 First prove that $f(0) = 0$ or $f(0) = 1$. If $f(0) = 0$, then $f(m) = 0$ for all m. If $f(0) = 1$, then $f(2m) =$

$2f(m)^2 - 1$. In this case $f(1) = 1$ leads to $f(m) = 1$ for all m. If $f(1) = 0$, then we get $f(n) = 0$ for odd n, $f(n) = -1$ if $n \equiv 2 \pmod 4$ and $f(n) = 1$ if $n \equiv 0 \pmod 4$; $f(1) = -1$ gives the solution $f(n) = -1$ for odd n and $f(n) = 1$ for even n. If $|f(1)| > 1$, then f becomes unbounded.

2.12 Show that $f(0) = 0$ or $f(0) = 1$. If $f(0) = 0$, then $f(1) = 0$ or 2. With $f(0) = 0$ and $f(1) = 0$, prove that $f(2) = 0$, $f(4) = 0$, $f(5) = 0$, etc. The ideas of problem 11 in chapter 2 are helpful here. Prove that $f(n) = 0$ for $0 \leq n \leq 10$. Use the identities in problem 11, chapter 2 to show that $f(n) = 0$ for all $n \geq 0$. If $f(0) = 0$ and $f(1) = 2$, the same method yields $f(n) = 2n$ for all $n \geq 0$. In the case $f(0) = 1$, prove that $f(1) = 1$ and continue to show that $f(n) = 1$ for all $n \geq 0$.

2.13 Write $a\mathbb{N}_0 + 1$ for the set $\{an + 1 : n \in \mathbb{N}_0\}$. Suppose such a function has been constructed. Consider $g : 3\mathbb{N}_0 + 1 \to 4\mathbb{N}_0 + 1$ defined by

$$g(n) = 4f\left(\frac{n-1}{3}\right) + 1.$$

This g is a bijection of $3\mathbb{N}_0 + 1$ on to $4\mathbb{N}_0 + 1$ and its inverse can be computed. Moreover g is multiplicative on $3\mathbb{N}_0 + 1$. Conversely we can start with a multiplicative bijection g of $3\mathbf{N}_0 + 1$ on to $4\mathbb{N}_0 + 1$ and construct f by

$$f(n) = \frac{g(3n+1) - 1}{4}.$$

Let P_1, P_2 be the sets of primes of the form $3n+1, 3n+2$ respectively. And let Q_1, Q_2 be the sets of primes of the form $4n + 1, 4n + 3$ respectively. Each of these is an infinite set. Choose a bijection h of $P_1 \cup P_2$ onto $Q_1 \cup Q_2$ that maps P_1 bijectively on to Q_1 and P_2 bijectively on to Q_2. Set $g(1) = 1$. If $n > 1$ is in $3\mathbb{N}_0 + 1$, write $n = \prod p_j$ with

p_j either of the form $3n+1$ or of the form $3n+2$; define $g(n) = \prod h(p_j)$. Verify that g is well defined and has all required property.

2.14 Write (c) in the form

$$\frac{f(m,m+n)}{m(m+n)} = \frac{f(m,n)}{mn}.$$

Use Euclidean algorithm to compute $d = \gcd(m,n)$ and conclude that

$$\frac{f(m,n)}{mn} = \frac{1}{d}.$$

Thus $f(m,n) = \text{lcm}\ (m,n)$.

2.15 Show that $f(n+2)-f(n) = f(n+3)(f(n+4)-f(n+2))$. Use this to prove that

$$f(3)-f(1) = f(4)f(6)\cdots f(2n+2)\{f(2n+3)-f(2n+1)\},$$

$$f(4)-f(2) = f(5)f(7)\cdots f(2n+3)\{f(2n+4)-f(2n+2)\}.$$

In the case $f(1) > f(3)$, this leads to an infinite strictly decreasing sequence of natural numbers. Hence $f(1) \le f(3)$. If $f(1) = f(3)$, prove that $f(2n+1) = f(1)$ for all n and $f(4)-f(2) = (f(1))^n\{f(2n+4)-f(2n+2)\}$. In the case $f(1) = 1$, this leads to $f(2n) = f(2)+(n-1)p$ and hence to the solution: $f(n) = 1$ for odd n and $f(n) = f(2)+((n/2)-1)p$ for even n. In the case $f(1) > 1$, we get $f(2n) = f(2)$. We get two solutions here, one corresponding to $f(1) = 2$ and $f(2) = k+1$ and the other corresponding to $f(2) = 2$ and $f(1) = k+1$. Thus $f(n) = 2$ for odd n and $f(n) = k+1$ for even n in the first case where as in the second case we get the solution $f(n) = k+1$ for odd n and $f(n) = 2$ for even n. If $f(3) > f(1)$, show that $f(2n-1) < f(2n+1)$ and hence $f(2n) = f(4)$ for all n. Prove that $f(3)-f(1) = (f(2))^n\{f(2n+3)-f(2n+1)\}$. Conclude that $f(2) = 1$ and hence $f(3)-f(1) = k$. We get the solution $f(n) = 1$ for n even and $f(n) = f(1)+((n-1)/2)p$ for n odd.

2.16 Evaluate $f(n)$ for some small values of n. Check that for these values of n, we can obtain $f(n)$ by first writing n in base 3, replacing 2 by 1 and 1 by 2 in this expansion and then converting the resulting string(which is in base 3) to base 10. Prove that this remains true for any n using induction. When does $f(n) = 2n$ hold good? Ans: 127.

2.17 Show first that $f(0) = 0$ or $f(n) = 1$ for all $n \in \mathbb{Z}$. Consider non-constant solution of the given equation. Show that $f(kp) = 0$ for all integers k. Using Fermat's little theorem, prove that $f(m) = f(m)^p$ for each integer m. Thus $f(m) = 0$ or $f(m) = \pm 1$. Choose $m = a$, a primitive root with respect to p. Then $f(a) \neq 0$. Consider the cases $f(a) = 1$ and $f(a) = -1$ separately. Ans: $f(n) \equiv 0$, $f(n) \equiv 1$,

$$f(n) = \begin{cases} 0 & \text{if } p|n, \\ 1 & \text{if } p \not| n, \end{cases}$$

and

$$f(n) = \begin{cases} 0 & \text{if } p|n, \\ 1 & \text{if } p \not| n,\ n \text{ is a square} \\ -1 & \text{if } p \not| n,\ n \text{ is not a square .} \end{cases}$$

The last function is precisely Legendre's symbol.

2.18 Prove first that $f\big(f(m)\big) = mf(1)^2$ and using this show that $f(1)f(n^2) = f(n)^2$. This shows that every prime dividing $f(1)$ also divides $f(n)$ for all $n \geq 1$. Consider the function $g(n) = f(n)/f(1)$. Show that g is a multiplicative function on $\mathbb{N}$ and $g\big(g(n)\big) = n$ for all n. Thus g takes primes to primes. Using $1998 = 2 \cdot 3^3 \cdot 37$, define g suitably to get the minimal value; remember g takes primes to primes and $g\big(g(p)\big) = p$. Ans: $f(1998) = 120$.

2.19 Show that $f(0) = 0$ and hence $f(-x) = -f(x)$ for all $x \in \mathbb{Z}$. Prove that $f(1) = -1, 0$ or 1 and hence $f(2) = 2f(1)$, $f(3) = 3f(1)$. For $x > 3$ prove that x^3 is a sum of five cubes each has absolute value smaller than x, using the identity

$$(2k+1)^3 = (2k-1)^3 + (k+4)^3 + (4-k)^3 + (-5)^3 + (-1)^3.$$

Using this representation, prove that $f(x) = xf(1)$. Thus $f(x) = -x$, $f(x) \equiv 0$ or $f(x) = x$.

2.20 Show that $f(1) = 1$ and $g(1) = 2$. Suppose $f(n) = k$ for some n. Show that the disjoint sets $\{f(1), f(2), \ldots, f(k)\}$ and $\{g(1), g(2), \ldots, g(n)\}$ together exhaust all the numbers from 1 to $g(n)$. Conclude that $g(n) = k+n$. Prove that $f(k) = k+n-1$. Show also that no two consecutive integers lie in the set $\{g(m) : m \in \mathbb{N}\}$. Conclude that $f(k+1) = k+n+1$. Use these three implications to get $f(240) = 388$.

2.21 Using the given relation, prove that $f(3n) \geq n$ for all natural numbers n. Moreover if $f(3k) > k$ for some k, show that $f(3n) > n$ for all $n \geq k$. Using $f(9999) = 3333$, conclude that $f(3n) = n$ for all $n \leq 3333$. Use this to show that $f(1982) = 660$.

2.22 Show that f is one-one and $f(0) = 0$. This implies that $f\big(f(m)^2\big) = m^2$. Prove that $f(0) = 0$ and hence $f\big(m^2+n^2\big) = f(m)^2 + f(n)^2$. Show that $f(1) = 0$ or 1 and $f(1) = 0$ leads to a contradiction. Use induction to prove that $f(n) = n$ for all n.

2.23 First show that $f(0) = 1$ and $f(-1) = 2$. Taking $n = -1$, show that f is an *even* function. Use Induction. Ans: $f(n) = n^2 + 1$.

2.24 Using (b), get an expression for $f(x^2)$ and hence for $f(x^4)$. Using $x^4 = x \cdot x^3, x^3 = x \cdot x^2$, get another expression for $g(x^4)$. Show that $k = 0$ or -1. Using prime decomposition, define f suitably for these values of k.

2.25 There are two constant solutions: $f(n) \equiv 0$ and $f(n) \equiv 2$. Assume f is not constant. Show that $f(0) = 1$, $f(-1) = 0$ and $f(-2) = -1$. Prove also that $f(-3) = -f(1)$ and $f(m) - f(-m) = f(2m-1)$. This gives $f(3) = 1 + f(2)$. Express $f(5)$ in two different ways to get a relation for $\lambda = f(1)$. Show that $\lambda = 0, -1$ or 2. If $\lambda = 0$, show that

$$f(4m) = 1, \quad f(4m+2) = -1, \quad f(2m+1) = 0, \quad \text{for all } m \in \mathbb{Z}.$$

Similarly $\lambda = -1$ leads to

$$f(3m) = 1, \quad f(3m+1) = -1, \quad f(3m+2) = 0, \quad \text{for all } m \in \mathbb{Z}.$$

Finally $\lambda = 2$ gives

$$f(m) = m + 1, \quad \text{for all } m \in \mathbb{Z}.$$

2.26 First show that $f(-1) = 0$ or $f(1) = 1$. If $f(-1) = 0$, prove that $f(-2) = 1 - f(1)$ and $f(-2)(1 - f(1)) = 0$. Thus $f(1) = 0$ or $f(1) = 2$. If $f(1) = 0$, then f is an *even* function and get

$$f(2m) = 1, \quad f(2m+1) = 0, \quad \text{for all } m \in \mathbb{Z}.$$

If $f(1) = 2$, prove that

$$f(m) = m + 1, \quad \text{for all } m \in \mathbb{Z}.$$

Finally $f(1) = 1$ leads to

$$f(m) = 1, \quad \text{for all } m \in \mathbb{Z}.$$

3 Equations on Real Line

3.1 Ans: (i) $f(x) = x+1$; (ii) $f(x) = x$; (iii) $f(x) = \pm e^{x-2}$; (iv) $f(x) = x^2$.

3.2 No, there are no such functions. Show that such a function satisfies $f(x) + f(x+1) = 1$. Use this to arrive at a contradiction.

3.3 Show that $xf(x)$ is constant and hence $f(x) = 0$ for all x.

3.4 Ans: $f(x) = (x^3 - x + 1)/2x(x-1)$.

3.5 Show first that $f(0) = 0$ or $f(0)^2 = 2$. Show that $f(0)^2 = 2$ leads to a contradiction using the given equation. Conclude that $f\big(f(x)\big) = x$. Using this prove the multiplicativity of f. Show that $f(1) = 0$ or $f(1) = 1$. In the former case $f(x) \equiv 0$ and this is not a solution. In the latter case show that f is also additive and hence $f(x) = x$.

3.6 Show that $f(0) = 0$ and $f\big(f(x)\big) = xf(1)$. Changing x to $f(x)$, show that $f(1) = 0$ implies that $f(x) \equiv 0$. If $f(1) \neq 0$, show that f is an onto function. Use this and choose r such that $f(r) = 1/f(1)$. Prove that $r = 1$ and hence $f(1) = \pm 1$. Show that $f(1) = -1$ gives an *odd* function f. But $f(-1) = f\big(f(1)\big) = f(1) = -1$, a contradiction. Thus $f(1) = 1$ and this leads to $f\big(f(x)\big) = x$. Conclude that f is additive and multiplicative. Thus $f(x) = x$ for all x.

3.7 Show that the equation implies that $f(0) = 0$ and hence $f(f(x)) = x$ for all x. Use this to conclude that $f(1) = 1$ or f is a constant function. Rule out constant solutions for the equation. Prove that $f(x) = x$ for all $x \in \mathbb{R}$.

3.8 Take any a, b with $b \neq 0$. Then $a + b = (ab^{-1}z + z)z^{-1}b$. Use multiplicativity and additivity with respect to z to conclude that $f(a+b) = f(a)+f(b)$. The multiplicativity also implies that $f(0) = 0$ or 1. If $f(0) = 1$ then conclude that $f(x) = 1$ for all $x \in \mathbb{R}$. But this contradicts the second property. With $f(0) = 0$, show that $f(x) = 0$ for all x or $f(x) = x$ for all x.

3.9 Show that $f(z) = 0$ if and only if $z \geq 2$. The given condition implies that $x \geq 2 - y$ if and only if $x \geq 2/f(y)$. Thus $f(z) = 2/(2 - z)$ for $0 \leq z < 2$.

3.10 Change z to $1-z$ and get $f(z)(z^2-z+1) = z^2-z+1$. Thus $f(z) = 1$ if $z \neq w_1, w_2$ where w_1 and w_2 are roots of $z^2 - z + 1 = 0$. Let $f(w_1) = \alpha$. Then $w_2 = 1 - w_1$ and hence $f(w_1)+w_1 f(w_2) = 1+w_1$. This gives $f(w_2) = w_2+1-\alpha w_2$. For each α, we get one function defined by $f(z) = 1$ for all $z \neq w_1, w_2$, $f(w_1) = \alpha$, and $f(w_2) = w_2 + 1 - \alpha w_2$.

3.11 For $r = p/q \in \mathbb{Q}^+$, use induction on $p+q$. The answer is $f(x) = 1/(1 + x)$ for $x \in \mathbb{Q}^+$.

3.12 Replace $x+y$ by x and $x-y$ by y. The equation takes the form $yf(x) - xf(y) = xy(x^2 - y^2)$. If we take $h(x) = f(x) - x^3$, then $yh(x) - xh(y) = 0$ and hence $h(x) = kx$. Thus $f(x) = x^3 + kx$.

OR Take $g(x) = f(x)/x$ and show that $(g(a+h)-g(a))/h = 2a + h$ for all $h \neq 0$. Conclude that g is differentiable and $g'(a) = 2a$. Solve the differential equation to get the answer.

3.13 Prove first that f is multiplicative and hence $f(1/x) = 1/f(x)$ for all $x \in (0, \infty)$. From this conclude that f is strictly decreasing on $(0, \infty)$. Thus f can have at most one fixed point in $(0, \infty)$. Since $f(1) = 1$, 1 is the unique fixed point of f in $(0, \infty)$. However $f(xf(x)) = xf(x)$ for all x in $(0, \infty)$. Obtain $f(x) = 1/x$ on $(0, \infty)$.

3.14 Fix $a \in \mathbb{R}_0$ and set $b_n = f^n(a)$, where $f^n(x) = f\big(f^{n-1}(x)\big)$. Obtain $b_{n+2} + b_{n+1} = 12b_n$ for $n \geq 1$ and $b_2 + b_1 = 12a$. Thus $b_n = c_1(-4)^n + c_2 3^n$ for some constants c_1 and c_2. Look at the behavior of $(-4/3)^n$ for odd and even n. Conclude that $b_n/3^n$ takes both positive and negative values as n becomes large. Force $c_1 = 0$ to get $b_n = c_2 3^n$. Get $c_2 = a$ by initial behavior of b_n. The solution is $f(x) = 3x$.

3.15 Observe that $f(0) = 0$ and $f(x^2) = f(-x^2)$. For $a > 0, b > 0$, the system of equations $x^2 - y^2 = a$, $2xy = b$ always has a real solution; find it. Conclude that $f(a) + f(b) = f\big(\sqrt{a^2 + b^2}\big)$ for all $a > 0, b > 0$. The function $g(x) = f\big(\sqrt{x}\big)$ is additive and nondecreasing on the set of all positive reals. Use this to conclude that $f(x) = cx^2$ for some positive constant c.

3.16 Put $f(0) = a$ and $g(0) = b$. Then $f(b + y) = g\big(f(y)\big)$ and $g(a+x) = f\big(g(x)\big)$. Use these and the second condition to get $g\big(g(a + x)\big) = g\big(x + f(b)\big)$. Use the injectivity of g. Conclude that $g(x) = x - a + f(b)$ and use this to get $f(x) = x + a$, and $g(x) = x + b$.

3.17 Replace x by $a+b$ and y by $a-b$ to prove that $f(b) + f(-b) = 2f(0)$. Using the equation, obtain the relation

$$\frac{f(a) - f(0)}{a} = \frac{f(a+b) - f(b)}{b}.$$

This shows that $\big(f(x) - f(0)\big)/x$ is a constant function for $x \neq 0$. This leads to $f(x) = cx + d$ for some constants c and d.

3.18 Replace x by $1/x$ and eliminate $f(1/x)$ from two relations. This can be done if $\alpha^2 \neq 1$. Ans:

$$f(x) = \frac{x(1 - \alpha x)}{(x+1)(1 - \alpha^2)},$$

for $\alpha^2 \neq 1$ and no solution if $\alpha^2 = 1$.

3.19 Replace z by $\omega z + a$ and again z by $\omega^2 z + \omega a + a$. Use $\omega^3 = 1$ and $\omega^2 + \omega + 1 = 0$. Ans:

$$f(z) = \frac{1}{2}\Big\{g(z) - g(\omega z + a) + g\big(\omega^2 z + \omega a + a\big)\Big\}.$$

3.20 If $f(x_0) = 0$ for some x_0, the $f(x) \equiv 0$. Assume $f(x)$ is never zero. Using $x + (y + z) = (x + y) + z$, get two expressions for $f(x + y + z)$. Conclude that f is constant. Hence $f(x) \equiv 0$; $f(x) \equiv 1$; and $f(x) \equiv -1$.

3.21 First prove that $f(0) = 0$. Methods employed in problem (6) of chapter 3 may be useful here. Use this information to prove that, $f\big(f(x)^2\big) = x^2$ and $f(x + y) = f(x) + f(y)$, for all reals x and $y \geq 0$. Prove that $f(1) = 1$ and $f(1/2) = 1/2$. Prove that $f\big(2f(x)f(y)\big) = 2xy$. Conclude that $f\big(f(x)\big) = x$. This leads to $f(x) = x$ for all reals x.

3.22 Show that $f(0) = 0$ and $f\big(y^n\big) = f(y)^n$. For any $x \in \mathbb{R}$ and $z \geq 0$, prove that $f(x + z) = f(x) + f(z)$. Use this to prove that f is additive; $f(x+y) = f(x)+f(y)$, for all reals x, y. Conclude that $f(rx) = rf(x)$ for all rationals r and reals x. Consider $f\big((x+r)^n\big)$ and use the binomial expansion and linearity of $f(x)$ to prove that $f\big(x^{n-k}\big) = f(1)^k f(x)^{n-k}$, for all $k \neq n$. Use this with appropriate k to conclude that f is either increasing or decreasing on $\mathbb{R}$. Thus $f(x) = x$ and $f(x) = -x$ are the only functions which satisfy the conditions of the given problem.

3.23 Replace x by $x+y$ and y by $x-y$ to get two relations and use them to obtain

$$f(x + 2y) - f(x - 2y) = f(y)\big[f(x + y) + f(x - y)\big].$$

Replacing y by $2y$ in the given equation get

$$f(x + 2y) - f(x - 2y) = f(x)f(2y).$$

Prove that $f(x) = 0$ for all x.

3.24 Observe that $f(x,y) = f(0,y-x)+x$. Hence $f(x,y) = (1-q)x + qy$ for some real q.

3.25 If f is constant, then $f(x) \equiv 0$ and $g(x) \equiv 0$. Assume f is not a constant function. Replace x by $g(x)$ in the equation and observe that the left hand side becomes symmetric in x, y. Get a relation using this symmetry. Prove that $f\big(g(y)\big) = \alpha y + \beta$ for some constants α and β. Use this to get an expression for $g\big(g(x)\big)$. Use the symmetry relation to conclude that

$$\big(g(x)-\beta\big)\big(f(y)+\alpha\big) = \big(g(y)-\beta\big)\big(f(x)+\alpha\big).$$

Use this to conclude that $f(x) = ax + b$ and $g(x) = cx + d$.

3.26 If $f(0) = c$, prove that $c^2 = f(x)f(-x) + x^2$. This forces $f(c) = 0$ or $f(-c) = 0$. Show that both lead to $c = 0$. Conclude that $f\big(f(x)\big) = f(x)$ and hence $f(x)f(y) = xy$. Show that $f(x) = x$ is the only possibility.

3.27 Ans: $f(x) = x$.

3.28 The first part is straight forward computation using the functional relation. The smallest possible value of m is 3. Show that $m = 1$ gives via $f(n+4m) = f(n)$ the absurd conclusion that $f(1)^2 = -1$. Similarly show that $m = 2$ forces $f(2)^2 = -1$. Show that $m = 3$ works. Make sure $f(n) \neq -1$ in this case so that the denominator never vanishes.

3.29 Show that the given conditions also imply that $f(x-19) \geq f(x) - 19$ and $f(x-94) \leq f(x) - 94$. using induction prove that

$$f(x+19n) \leq f(x) + 19n, \quad \text{and} \quad f(x+94n) \geq f(x) + 94n,$$

$$f(x-19n) \geq f(x) - 19n, \quad \text{and} \quad f(x-94n) \leq f(x) - 94n.$$

Use $1 = 5 \times 19 - 94$ to conclude that $f(x+1) \le f(x)+1$. Similarly use $1 = 18\times 94 - 89\times 19$ to conclude that $f(x+1) \ge f(x)+1$.

3.30 Replace x by $-1/x$, and get another relation. Combine this with original relation. Ans: $f(x) = (1+x^3)/2x$.

3.31 Show that $f(0) = f(1) = 1/2$. Use this to prove that $f(x) = 1/2$ for all $x \in \mathbb{R}$.

3.32 Prove that $f(0) = 1$. Show that f also satisfies

$$f(xz+yz) - 1 + f(z)\big[f(xy) - 1\big] = f(xy+xz) - 1 + f(x)\big[f(yz) - 1\big].$$

Use this and the given relation to obtain

$$f(xy)\big[f(1) - 1 - f(x)\big] = f(1) - f(x^2y) - f(x).$$

This implies that $f(1)\big(f(1)-2\big) = f(x)\big(f(1)-2\big)$. If $f(x) \not\equiv 1$, then $f(1) = 2$. Using this value of $f(1)$, show that $f(xy+x) = f(xy)+f(x)-1$. If we use the equation to get $f(xy+x)$, we finally obtain

$$f(xy)f(x) - f(x^2y) + 1 = f(xy) + f(x) - 1.$$

Show that the function $F(x) = f(x) - 1$ is both additive and multiplicative. Use this property to find F and hence f. Ans: $f(x) \equiv 1$ and $f(x) = x + 1$.

3.33 Use induction and show that $f(n) = 1$ for all natural numbers. Suppose $f(r) = 0$ for some rational $r > 1$. Define a new function $g(x) = 1 - f\Big(\big(r - [r]\big)x + [r]\Big)$. Show that $g : \mathbb{Q} \to \{0, 1\}$ also has the same property as that of f. Thus $g(n) = 1$ for all natural numbers. Show that $g(q) = 0$ where q is the denominator of r, and get a contradiction.

3.34 Write the equations in the form $xf(y) - yf(x) = (x - y)g(x + y)$ which is valid even if $x = y$. Put $x = 0$ to get $g(x) = f(0) = \beta$ for all $x \neq 0$. Thus $xf(y) - yf(x) = \beta(x - y)$ for all x, y such that $x + y \neq 0$. Take $x = 1$ and $y \neq -1$ to get $f(y) = \alpha y + \beta$, where $\alpha = f(1) - \beta$. Taking $y = 2$, get $f(2) = 2f(1) - \beta$. Again $x = -1$ and $y = 2$ in the original equation gives $f(-1) = -\alpha + \beta$. Thus $f(y) = \alpha y + \beta$ for all y. Similarly $x = 1$ and $y = -1$ in the first equation also gives $g(0) = \beta$. The solution is therefore: $f(x) = \alpha x + \beta$ and $g(x) = \beta$.

3.35 We show that $g(x) = f(x)$. Interchanging x and y, we see that $xf(y) - yg(x) = xg(y) - yf(x)$ for all $x \neq y$.Thus

$$\frac{f(x) - g(x)}{x} = \frac{f(y) - g(y)}{y}, \quad x, y \text{ non-zero }, x \neq y.$$

Take any fixed non-zero real number λ and set $a = \big(g(\lambda) - f(\lambda)\big)/\lambda$. Then $f(x) = g(x) + ax$ for all $x \in \mathbb{R} \setminus \{0, \lambda\}$. Take suitable x, y in the given equation to conclude that $a = 0$. Thus $f(x) = g(x)$ for all $x \neq 0$. Take $x = 0$, $y = \lambda$ in the given equation to get $f(0) = h(\lambda)$. Similarly prove that $h(\lambda) = g(0)$. Thus $f(x) = g(x)$ is valid for all real x. Previous problem is applicable. Ans: $f(x) = g(x) = ax + b$, $h(x) = b$.

4 Cauchy's Equation

4.1 Express both h and g in terms of f and use this in the equation to get Cauchy's type equation for f. Ans: $f(x) = \alpha x + a + b$, $g(x) = \alpha x + a$ and $h(x) = \alpha x + b$ for some constants α, a, b.

4.2 First express both f and g in terms of h and k. Use these expressions to get a relation involving only h and k. Using new functions $H(x) = h(x) - h(0)$ and $K(x) = k(x) - k(0)$, show that $H(x)+H(-x) = K(x)+K(-x)$. Decompose $h(x) = h_e(x) + h_o(x)$ and $k(x) = k_e(x) + k_o(x)$ as sum of *even* and *odd* parts. Show that $h_e(x+y)+h_e(x-y)+2k(0) = 2h_e(x) + 2k_e(y)$. Use this and the relation $H_e(x) = K_e(x)$ proved earlier to get an equation $H_e(x+y) + H_e(x-y) = 2H_e(x) + 2H_e(y)$. The general solution of this is (remember that H_e is *even*) given by $H_e(x) = \alpha x^2$. This determines $h_e(x)$ and $k_e(x)$. Show also that h_o satisfies the relation $h_o(x+y) + h_o(x-y) = 2h_o(x)$. The continuous solution of this is $h_o(x) = \beta x$. Use this to get a relation for k_o in the form $k_o(x+y) + k_o(x-y) = 2k_o(x)$. This has general continuous solution $k_o(x) = \gamma x$. This completely determines the solutions of the given equation:

$$\begin{aligned} f(x) &= \alpha x^2 + (\beta+\gamma)x + 2a + 2b - c, \\ g(x) &= \alpha x^2 + (\beta-\gamma)x + c, \\ h(x) &= \alpha x^2 + \beta x + a, \quad k(x) = \alpha x^2 + \gamma x + b. \end{aligned}$$

4.3 We may start with the assumption that neither $h(x) \equiv 0$ nor $k(x) \equiv 0$. Otherwise we get constant functions for f and g with $f(x) = -g(x)$. It may also be assumed that $h(a) \neq 0$ and $g(a) \neq 0$ for some a. Replace y by $y+a$ and introduce new functions

$$F(t) = \frac{f(t+a)}{k(a)}, \quad G(t) = \frac{g(t-a)}{k(a)}, \quad K(t) = \frac{k(t+a)}{k(a)},$$

and get the equation in F, G, h, K. Observe $K(0) = 1$. Express both F and G in terms of K and h. Use this to eliminate F and G and get an equation

$$h(x+y)K(x-y) = h(x)K(x) + h(y)K(-y) + \mu,$$

for some constant μ. Use this to get a relation $h(2x) = h(x)K(x) + h(x)K(-x) + \mu$. Prove further that

$$h(x+y)\{K(x-y) + K(x+y)\} = h(2x) + h(2y).$$

This leads to

$$h(x+y) + h(x-y) = 2h(x)K_e(y),$$

where K_e is the *even* part of K. [See problem 9 of Chapter 4.] This determines h and K_e. Use the equation again to show that the *odd* parts satisfy

$$h_o(x+y)K_o(x-y) = h_o(x)K_o(x) - h_o(y)K_o(y).$$

This implies that $h_o(x+y)K_o(x-y) = h_o(x-y)K_o(x-y)$. If $h_o(x) \not\equiv 0$, use the known form of h got earlier to solve for K_o. Otherwise we get that h is *even*. Use $h(a) \neq 0$ to show that the function

$$H(t) = \frac{h(t+a) + h(t-a)}{2h(a)}$$

satisfies the equation $K(y+x) + K(y-x) = 2K(y)H(x)$. Check that H is *even* and use this to prove that $K_o(x+y) + K_o(x-y) = 2K_o(x)H(y)$. The solutions of this are known. This determines K_o and hence K. Trace back earlier functions.

4.4 If $f_j(x) = f(0, 0, \ldots, 0, x, 0, \ldots, 0)$, where we retain j-th coordinate and set all other coordinates equal to 0, then f_j satisfies Cauchy's equation and hence it is linear. Ans: $f(x_1, x_2, \ldots, x_n) = c_1x_1 + c_2x_2 + \cdots + c_nx_n$ for some constants $c_1, c_2, \ldots, c_n$.

4.5 Show that $f(0) = 0$ or $f(0)^{2001} = 1$. In the latter case consider $g(x) = f(x)/f(0)$. Prove that $g(x+y) = g(x)g(y)$. Conclude that $g(x) = e^{bx}$ for some complex number b. Use the second condition to prove that b is real. Ans: $f(x) \equiv 0$ or $f(x) = ae^{bx}$, where b is real and a is a 2001-th root of unity (i.e., $a^{2001} = 1$).

4.6 Assume $g(x) \not\equiv 0$ and $h(x) \not\equiv 0$. Show that h is an *odd* function. Split f and g in to their *odd* and *even* parts: $f(x) = f_o(x) + f_e(x)$ and $g(x) = g_o(x) + g_e(x)$. Deduce that $f_e(x+y) - f_e(x-y) = 2h(y)g_o(x)$ and $f_o(x+y) - f_o(x-y) = 2h(y)g_e(x)$. Using the equation for *even* part of f, show that $g_o(x) = ch(x)$ for some constant c. Thus the equation for f_e reads as $f_e(x+y) - f_e(x-y) = 2ch(x)h(y)$. Compare this with problem 14 in chapter 4 and solve for f_e and h. Observe that $f_o(y) = g_e(0)h(y)$ and this determines f. Use this expression for $f_o(x)$ in the equation satisfied by f_o and get $g_e(0)\big[h(x+y) - h(x-y)\big] = 2h(y)g_e(x)$. Since h has been already determined, this gives g_e. Thus both g_e and g_o are expressed using h. This gives g and the complete solutions of the equation. Ans:

(a) $f(x) \equiv \lambda$, $h(x) \equiv 0$, and $g(x)$ arbitrary continuous function;

(b) $f(x) \equiv \lambda$, $h(x)$ arbitrary continuous function, and $g(x) \equiv 0$;

(c) $f(x) = \dfrac{\alpha^2}{2}x^2 + \dfrac{\alpha\gamma}{c}x + k$, $h(x) = \dfrac{\alpha}{c}x$, and $g(x) = \gamma + \alpha c\, x$;

(d) $f(x) = \alpha^2 \cos\beta x + \dfrac{\alpha\gamma}{c}\sin\beta x + k$, $h(x) = \dfrac{\alpha}{c}\sin\beta x$ and $g(x) = \gamma\cos\beta x + \alpha c \sin\beta x$;

(e) $f(x) = \alpha^2 \cosh\beta x + \dfrac{\alpha\gamma}{c}\sinh\beta x + k$, $h(x) = \dfrac{\alpha}{c}\sinh\beta x$ and $g(x) = \gamma\cosh\beta x + \alpha c \sinh\beta x$.

4.7 First show that $f(0) = 0$. Using this information, prove that f_o satisfies $f_o(x+y) + f_o(x-y) = 2f_o(x)$ and f_e satisfies $f_e(x+y) + f_e(x-y) = 2f_e(x) + 2f_e(y)$. Ans: $f(x) = \alpha x + \beta x^2$.

4.8 Show first that $3f(0) = 3g(0) + h(0)$. Using $F(x) = f(x) - f(0), G(x) = g(x) - g(0)$ and $H(x) = h(x) - h(0)$, get an equation in F, G, H. Show that $2F(x) = G(x) + H(x)$. Use this to eliminate F and prove that $\phi(x) = G(x) - H(x)$ satisfies Cauchy's equation. By continuity $\phi(x) = \alpha x$. Use this to show that H satisfies an equation $H(x+y)+H(y+z)+H(z+x) = H(x)+H(y)+H(z)+H(x+y+z)$. Use previous problem. Ans: $h(x) = a+\beta x+\gamma x^2$, $g(x) = b+(\alpha+\beta)x+\gamma x^2$ and $f(x) = c + \big((\alpha/2) + \beta\big)x + \gamma x^2$, where a, b, c satisfy $3c = 3b + a$.

4.9 First express g in terms of h and f and use this in the given equation to get an equation in two unknown functions h and f:

$$f(x + y) + h(xy) + h(1) - f(xy + 1) = h(x) + h(y).$$

Using this, prove that

$$f(xy + z) + f(x + y) - f(xy + 1) \\ = f(x + yz) + f(y + z) - f(yz + 1).$$

Using the continuity, let $z \to 0$ and get

$$f(xy) + f(x + y) - f(xy + 1) = f(x) + f(y) - f(1).$$

Compare this with problem 11 of chapter 4. After obtaining f, put this in the relation satisfied by f and h to obtain an equation for h. Using this equation for h show that a suitable variant $\tilde{h}$ of h satisfies the equation $\tilde{h}(xy) = \tilde{h}(x) + \tilde{h}(y)$. This determines $\tilde{h}$ and hence h. Finally compute g. Ans:

$$g(x) = -\alpha x + \lambda \ln x + \gamma + 2h(1);$$
$$h(x) = \lambda \ln x + \left(\frac{\alpha}{2}\right)x^2 + \beta x + \gamma + h(1);$$
$$f(x) = \left(\frac{\alpha}{2}\right)x^2 + \beta x + \gamma + f(1),$$

where $\left(\frac{\alpha}{2}\right) + \beta + \gamma = 0$.

4.10 Change x to $-x$ and y to $-y$ in the given equation and use the resulting expression in conjunction with the original equation to obtain

$$f(x+y) - f(-x-y) + f(x)f(y) - f(-x)f(-y) = 0.$$

Use this to prove that the *odd* part f_o and the *even* part f_e of f satisfy the equation

$$f_o(x+y) + f_o(x-y) + 2f_o(x)f_e(y) = 0.$$

Compare this with the problem 9 of chapter 4. The only possible solutions are $f_o(x) = \alpha x$, $f_e(x) \equiv -1$ and $f_o(x) \equiv 0$, $f_e(x)$ arbitrary. In the first case we get $f(x) = \alpha x - 1$ and the given relation shows that $\alpha = 1$. In the second case we have to work with an *even* function f satisfying the given equation. Using the *evenness* of f prove that

$$f(x+y) - f(x-y) = f(xy+1) - f(xy-1).$$

It is sufficient to find $f(x)$ for nonnegative values of x. Use the above equation to prove

$$f(uv+uw) - f(uv-uw) = f(uvw+u) - f(uvw-u),$$

for all nonnegative values of u, v, w. Conclude that

$$f(x+y) - f(x-y) = f\left(2\sqrt{xy}\right) + 1,$$

for all $x > 0, y > 0$. Use this to prove that the function $g(x) = f\left(\sqrt{x}\right) + 1$ is additive on positive reals. Conclude that $f(x) = \alpha x^2 - 1$. Use the equation to show that $\alpha = 1$. Ans: $f(x) = x - 1$ and $f(x) = x^2 - 1$.

4.11 First show that $f(0) = 0$ and $f(1) = 2$. Prove that f is an *even* function. Obtain the relation

$$f(x+y) - f(x-y) = f(xy+1) - f(xy-1).$$

Proceed as in the previous problem. Alternately use the above relation with the original equation to prove that $2f(xy) = f(x)f(y)$. Use this to prove that $f(x+y)+f(x-y) = 2f(x)+2f(y)$. Solve this equation using the continuity of f. Ans: $f(x) = 2x^2$.

4.12 First express f in terms of h and use this to eliminate f from the given equation. Use this to express g in terms of h and eliminate g as well. You get

$$h(0)\big[h(x+y) - h(xy+1)\big] + h(1)h(xy) = h(x)h(y).$$

If $h(0) = 0$ and $h(1) = 0$, then $h(x) \equiv 0$. In this case $g(x) = \alpha$ and $f(x) = 1 - \alpha$ describe all the solutions. If $h(0) = 0$ and $h(1) \neq 0$, then $h(x)/h(1)$ is a multiplicative solution and hence $h(x) = \beta x^\mu$ for some μ. We obtain $f(x) = 1 - \alpha$ and $g(x) = \beta^2 x^\mu + \alpha$.

If $h(0) \neq 0$ and $h(1) = 0$, then $\phi(x) = -h(x)/h(0)$ satisfies $\phi(x+y) + \phi(x)\phi(y) = \phi(xy+1)$. Using exercise 10, get $h(x) = \alpha(x-1)$ or $h(x) = \alpha(x^2-1)$. Finally consider the case $h(0) \neq 0$ and $h(1) \neq 0$. The function $H(x) = h(x)/h(0)$ satisfies the equation

$$H(x+y) - H(xy+1) + H(1)H(xy) = H(x)H(y)$$

Show that $H_o(x)$, the *odd* part of H satisfies $H_o(x+y) + H_o(x-y) = 2H_o(x)H_o(y)$ and conclude that $H_o(x) \equiv 0$. Thus H is even. Prove that $H(x+y) - H(x-y) = H(xy+1) - H(xy-1)$ where $H(0) = 1$. Use this to prove that $H(xyz+x) + H(xyz-x) = H(xy+xz) - H(xy-xz)$. This reduces to $H(u+v) - H(u-v) = H(2\sqrt{uv}) + 1$. Compare this with exercise 15 of chapter 3. Show that this implies that $H(x) = \gamma x^2 + 1$. Ans:

(a) $f(x) = 1-\alpha,\ g(x) = \alpha,\ h(x) = 0$;

(b) $f(x) = 1-\alpha,\ g(x) = \beta^2 x^\mu + \alpha,\ h(x) = \beta x^\mu$;

(c) $f(x) = \beta^2(1-x) + 1 - \alpha$, $g(x) = \beta^2 x(x+2) + \alpha$, $h(x) = \beta(1-x)$;

(d) $f(x) = \beta^2(\gamma x^2 + 1)$; $g(x) = \beta^2(\gamma x - 1)^2 + \alpha - \beta^2$, $h(x) = \beta(\gamma x^2 + 1)$.

4.13 Use the techniques of previous exercise to get an equation only in h:

$$h(0)\big[h(x+y) - h(xy+1)\big] + h(1)h(xy) = h(x)h(y).$$

Solve this as in the previous exercise. Ans:

(a) $f(x) = \alpha$, $g(x) = \alpha$, $h(x) = 0$;

(b) $f(x) = \alpha$, $g(x) = \beta^2(x-1)^\mu + \alpha$, $h(x) = \beta x^\mu$;

(c) $f(x) = \alpha - \beta^2(1-x)$, $g(x) = \alpha + \beta^2 x$, $h(x) = \beta(1-x)$;

(d) $f(x) = \alpha - \beta^2(\gamma x^2 + 1)$, $g(x) = \alpha + \beta^2\gamma^2(x^2+1) - 2\beta^2\gamma((\gamma+1)x - 1)$, $h(x) = \beta(\gamma x^2 + 1)$.

4.14 Follow the method employed in the problem 11 of chapter 4. You end up with Cauchy's equation. Ans: $f(x) = x^2 + x$.

4.15 First show that if either $g(0) = 0$ or $h(0) = 0$, then the only solutions are: $f(x) \equiv 0$, $g(x) \equiv 0$ and $h(x)$ arbitrary or $f(x) \equiv 0$, $g(x)$ arbitrary and $h(x) \equiv 0$, respectively. Assume $\alpha = g(0) \neq 0$ and $\beta = h(0) \neq 0$. Show that $F(x) = f(x)/\alpha\beta$ satisfies $F(x+y) = F(x)F(y)$. Ans: $f(x) = \alpha\beta a^x$, $g(x) = \alpha a^x$, $h(x) = \beta a^x$, for some positive a.

4.16 Take $f(0) = \alpha$ and consider $g(x) = f(x) - \alpha$. Show that g is an odd function. Use this to prove that g is also additive. Conclude $g(x) = \beta x$, for some constant β.

4.17 Consider new functions g and h introduced by

$$f(\sqrt{t}) = g(t)e^{ih(t)}.$$

Then g and h are continuous and satisfy the equations

$$\begin{aligned} g(u+v) &= g(u)g(v), \\ h(u+v) &= h(u) + h(v) + 2k(u,v)\pi, \end{aligned}$$

where $k(u,v)$ is an integer valued continuous function for all positive reals u, v. Use the continuity of $k(u,v)$ to prove that it remains constant, say k, for all u, v. Conclude $g(t) = e^{\lambda t}$, $h(t) = \mu t - 2k\pi$. Construct f using g and h. Ans: $f(t) = e^{\xi t^2}$ for some complex number ξ.

4.18 Ans: $f(\xi) = \lambda\xi + \mu\bar{\xi}$

4.19 Show that $h(x) = f(x) + g(x)$. Eliminate $h(x)$ using this and get an equation involving only f and g. Use this to conclude that $f(x) = g(x) + \alpha$ for some constant α. Use this information to eliminate g and get an equation involving only f. This equation shows that $F(x) = f(x) - f(0)$ satisfies

$$F(x+y) + F(x-y) = 2F(x).$$

Show that $F(x)$ is an *odd* function. Interchanging x and y and using that $F(x)$ is an *odd* function, get

$$F(x+y) - F(x-y) = 2F(y).$$

These two imply that $F(x)$ is an additive function. Thus the solutions are:

$$f(x) = F(x) + \beta, \quad , g(x) = F(x) + \gamma, h(x) = 2F(x) + \beta + \gamma,$$

where $F(x)$ is an additive function.

5 Additional Hypothesis

5.1 Consider $Q(x) = P(x) - x^2$. Prove that $Q(x) \equiv c$ for some constant c. Ans: $P(x) = x^2 + c$.

5.2 Ans: $f(x) \equiv 0$ and $f(x) \equiv 1$.

5.3 Consider $g(x) = f(x) - x^3/3$. What is the equation satisfied by g? Ans: $f(x) = cx + \dfrac{x^3}{3}$.

5.4 Assume $P(x) \not\equiv 0$. Show that $P(0) = 0$ leads to identically vanishing polynomial. Conclude that $P(0) = 1$ and $P(x)$ has leading coefficient 1. If α is a root of $P(x) = 0$, then so is $2\alpha^3 + \alpha$. Show that every root of $P(x) = 0$ must lie on the unit circle. Thus $|\alpha| = 1$ and $|2\alpha^3 + \alpha| = 1$. Use this to conclude that $\alpha^2 = -1$. Ans: $P(x) \equiv 0$, $P(x) \equiv 1$, and $P(x) = (1 + x^2)^n$, where n is a natural number.

5.5 Suppose $\alpha \neq 0$ is a root of $P(x) = 0$. Show that $|\alpha| = 1$. Thus all roots of $P(x) = 0$ lie either at the origin or on the unit circle. Show that for any odd natural number q, $1 + x + x^2 + x^3 + \cdots + x^{q-1}$ satisfies the given equation. Write

$$P(x) = (-x)^l (1-x)^m (p_{n_1}(x))^{k_1} (p_{n_2}(x))^{k_2} \ldots (p_{n_r}(x))^{k_r} Q(x),$$

for some integer $r \geq 0$, where l is even, $n_1, n_2, \ldots, n_r$ are odd positive integers, $k_1, k_2, \ldots, k_r$ are nonnegative integers and $p_j(x) = 1 + x + x^2 + \cdots + x^{j-1}$ for $j = n_1, n_2, \ldots, n_r$. Assuming $Q(x) = 0$ has no roots at 0 or 1 and has no factors of th form $(p_j)^k$, prove that $Q(x)$ reduces to a constant. Use this to conclude that either $P(x) \equiv 0$ or

$$P(x) = (-x)^l (1 - x)^m (p_{n_1}(x))^{k_1} (p_{n_2}(x))^{k_2} \ldots (p_{n_r}(x))^{k_r}.$$

5.6 Consider $Q(x) = P(x) - x$. Show that $Q(x)$ is a constant polynomial. Ans: $P(x) = x + c$ for some constant c.

5.7 Put $y = x$ and conclude that if P is not constant, then $x = 0$ is a root of $P(x) = 0$. Set $P(x) = x^k Q(x)$ and get an equation for Q. What can you say about Q? Ans: $P(x) \equiv 0$, or $P(x) = x^k$ for some nonnegative integer k.

5.8 Write $f(x) = x^k P(x)/x^l Q(x)$, where $P(0) \neq 0$ and $Q(0) \neq 0$. If m and n are degrees of P and Q respectively, introduce $x^m P(1/x) = P_1(x)$ and $x^n Q(1/x) = Q_1(x)$. Show that $m - n = 2(l - k)$, $P_1(x) = P(x)$ and $Q_1(x) = Q(x)$. Use this to get the desired results. You may have to consider odd and even cases separately.

5.9 First show that every positive real number is in the range of f. Hence there is some y_0 such that $f(y_0) = 1$. Taking $x = 1$ and $y = y_0$, conclude that $y_0 = 1$ and $f(1) = 1$. If a and b are fixed points of f show that so is ab. Use this to conclude that if a is any fixed point of f then $a \leq 1$. Since $xf(x)$ is a fixed point of f, it follows that $xf(x) \leq 1$. Using the equation, show that $1/xf(x)$ is also a fixed point of f. Conclude that $1/(xf(x)) \leq 1$. Thus $f(x) = 1/x$ is the only solution.

5.10 Put $h(x) = f(g^{-1}(x))$. Then h is a continuous bijection such that $h(x) + h^{-1}(x) = 2x$. Conclude that $h(x) = x + c$. Show that $c = 0$ and hence $f(x) = g(x)$ for all x.

5.11 The condition forces that degree of $P(x) = 3$. Write a general polynomial of degree 3 and deduce conditions on coefficients. Ans: $P(x) \equiv 0$, $P(x) = 12a^3 + 6a^2x + ax^2 + (1/18)x^3$ where a is a real number.

5.12 Use the substitution $x = e^u, y = e^v, z = e^w$ and $g(t) = f(e^t)$. What is the equation for $g(t)$? Use this equation and Jensen's equation to deduce $g(t) = \alpha t + \beta$. Show that $\beta = 0$ using the given equation. Conclude that $f(x) = \alpha \ln x$, for some $\alpha \in \mathbb{R}$.

5.13 Prove that $f(0) = f(1) = f(-1) = 0$ and $f(-x) = -f(x)$. Concentrate only on positive reals and effect the substitution $x = e^u, y = e^v$, $g(t) = e^{-t} f(e^t)$. Ans: $f(x) = \alpha x \ln |x|$.

5.14 Assume $P(x) \not\equiv 0$. Show that $P(0) = 0$. Prove that if $n \neq 1$, then $P(x) = 0$ has infinitely many zeros. Conclude that $n = 1$. Prove that $P(m) = mP(1)$ for all $m \in \mathbb{N}$. Consider $Q(x) = P(x) - P(1)x$. Show that $Q(m) = 0$ for all $m \in \mathbb{N}$. Conclude that $P(x) = P(1)x$.

5.15 Show that f is an *even* function, and use this to conclude that g is necessarily an *odd* function. Thus $g(0) = 0$, $f(0) = 1$, and $f(x)^2 + g(x)^2 = 1$ for all reals x. Changing x to $x - y$ and y to $-y$ get a relation

$$g(x - y)g(y) = g(y)\{g(x)f(y) - g(y)f(x)\}.$$

If $g(y_0) = 0$ for some y_0, prove that $g(x - y_0)^2 = g(x)^2$ and $f(y_0)^2 = 1$. If $f(y_0) = 1$, show that $g(x - y_0) = g(x)$. Similarly $f(y_0) = -1$ forces $g(x - y_0) = -g(x)$. In any case $g(x - y_0) = g(x)f(y_0) - g(y_0)f(x)$. If $g(y) \neq 0$ for any y, then $g(x - y) = g(x)f(y) - g(y)f(x)$.

5.16 Exclude trivial solutions $f(x) \equiv 0$, $f(x) \equiv 1$, and $f(x) \equiv -1$. If f is a solution so is $-f$. Note that f is non-negative throughout or non positive throughout. Assume $f(x) \geq 0$ and show that $f(0) = 1$. Show that if $f(x_0) = 0$ for some x_0, then $f(x) \equiv 0$. Consider $g(x) = \ln f(x)$. Get an equation for $g(x)$ and solve it. Ans: Apart from trivial

solutions listed earlier, other solutions are $f(x) = e^{ax^2}$ and $f(x) = -e^{ax^2}$, for some constant a.

5.17 Ans: $P(x) = (x-2)(x-4)(x-8)(x-16)$.

5.18 Choose first $x_1 = x_2 = (x+y)/2$, $x_3 = 1-x-y$ and $x_4 = x_5 = \cdots = x_n = 0$ and then $x_1 = x$, $x_2 = y$, $x_3 = 1-x-y$ and $x_4 = x_5 = \cdots = x_n = 0$. Using the relations so obtained, show that f satisfies Jensen's equation. Conclude that $f(x) = cx+d$, for some constants c and d. Ans: $f(x) = (1-nf(0))x + f(0)$.

5.19 Consider $g(x) = f(x) - f(0) - xf'(0)$. Show that g satisfies the differential equation $xg'(x) = 2g(x)$. Solve this equation. Ans: $f(x) = ax^2 + bx + c$.

5.20 Show by induction that $f(nx) = a^{\frac{(n^2-n)x^2}{2}} f(x)^n$ for all real numbers x and natural numbers n. Prove that $f(1) \geq 0$. If $f(1) = 0$, prove that $f(x) = 0$ for all $x \in \mathbb{R}$. If $f(1) > 0$, show that $f(x) = a^{\frac{x^2}{2}+cx}$, for all positive rationals x, where $c = -\frac{1}{2} + \log_a f(1)$. Use this and continuity of f to prove $f(x) = a^{\frac{x^2}{2}+cx}$, for all positive reals and hence also for all reals x.

5.21 Prove that $f(0) = 0$ and hence f is an *odd* function. Consider the functions g, h defined by

$$g(x) = \frac{1 - f\left(\frac{x}{2}\right)^2}{1 + f\left(\frac{x}{2}\right)^2}, \quad h(x) = \frac{2f\left(\frac{x}{2}\right)}{1 + f\left(\frac{x}{2}\right)^2}.$$

Show that $g(x-y) = g(x)g(y) + h(x)h(y)$. Thus $g(x) = \cos \alpha x$ and $h(x) = \sin \alpha x$. Ans: $f(x) = \tan \alpha x$.

5.22 Consider $g(x) = f(x)/(1+f(x))$. Ans: $f(x) = \alpha x/(1-\alpha x)$.

5.23 Ans: $f(x) = (ax+1)^{\alpha}$.

5.24 A suitable variant g of f satisfies $g(xy) = g(x)g(y)$. Ans: $f(x) = \alpha a^x$ for some $\alpha \geq 0$ and $a > 0$.

5.25 Ans: $f(x) = \left(\dfrac{K+x}{K-x}\right)^{\alpha}$.

6 Additional Problems

6.1 The only solutions are $f(x) = 0$ for all x; $f(x) = 1/2$ for all x; and $f(x) = x^2$ for all x.

Put $x = y = z = 0$ to get $f(0) = f(0)\big(f(0) + f(t)\big)$ for all t. In particular, $f(0) = 2f(0)^2$ so that $f(0) = 0$ or $f(0) = 1/2$. If $f(0) = 1/2$, then $f(t) = 1/2$ for all t.

Suppose $f(0) = 0$. Putting $z = t = 0$, we get $f(xy) = f(x)f(y)$ for all x, y. Hence f is multiplicative and $f(1)^2 = f(1)$. Thus $f(1) = 0$ or 1. If $f(1) = 0$, it is easy to check that $f(x) = 0$ for all x. Assuming $f(1) = 1$, taking $x = 0$, $y = t = 1$, we get $f(z) = f(-z)$ for all z. Using $f(x^2) = f(x)^2$ for all x and $f(z) = f(-z)$ for all z, infer that $f(x) \geq 0$ for all x.

Taking $t = x$, $z = y$, we get $f(x^2 + y^2) = \big(f(x) + f(y)\big)^2$. Thus $f(x^2 + y^2) \geq f(x)^2 = f(x^2)$. This shows that $f(u) \geq f(v)$ if $u \geq v \geq 0$. Hence f is an increasing function on the positive reals. Taking $y = z = t = 1$, we see that

$$\big(f(x-1)f(x+1) = 2\big(f(x)+1\big).$$

This implies that $f(n) = n^2$ for all non-negative integers n. Since f is even, $f(n) = n^2$ for all integers n. Using multiplicativity of f, we see that $f(r) = r^2$ for all rationals r. Since f is increasing on positive reals and $f(r) = r^2$ for all rationals, it is easy to show that $f(x) = x^2$ for all $x \geq 0$. Since f is even, we infer that $f(x) = x^2$ for all real x.

6.2 We show that the only non-zero solution of

$$f(x+y)+f(x)f(y)=f(xy)+f(x)+f(y) \qquad (1)$$

is $f(x)=x$. Replacing y by $y+z$ in (1) and using again (1) in the resulting relation, we obtain

$$\begin{aligned} &f(x+y+z)+f(x)f(y)+f(y)f(z)\\ &\quad+f(z)f(x)-f(x)f(y)f(z)-f(x)-f(y)-f(z)\\ &\qquad=f(x^2yz)-f(xy)f(xz)-f(x)f(yz). \end{aligned}$$

The symmetry of the left hand side now implies that

$$\begin{aligned} &f(x^2yz)-f(xy)f(xz)-f(x)f(yz)\\ &\qquad=f(xy^z)-f(xy)f(yz)-f(y)f(xz). \end{aligned} \qquad (2)$$

Taking $y=1$ in (2), we obtain

$$f(x^2z)=(1-f(1))f(xz)+f(x)f(xz).$$

However, we may write

$$f(x^2z)=f(x+xz)+f(x)f(xz)-f(x)-f(xz),$$

by (1). Thus we obtain

$$f(x+xz)=(2-f(1))f(xz)+f(x). \qquad (3)$$

Taking $y=0$ in (1), we get $(f(x)-2)f(0)=0$. This shows that $f(0)=0$ or $f(x)\equiv 2$. It is easy to check that $f(x)\equiv 2$ is a solution of (1). We may assume therefore that $f(0)=0$. Taking $x=y=2$ in (1), we also get $f(2)^2=2f(2)$. Hence $f(2)=0$ or $f(2)=2$, We consider these cases separately.

Case 1. Suppose $f(2)=0$. Taking $x=y=1$ in (1), we obtain $f(2)+f(1)^2=3f(1)$. Since $f(2)=0$, this implies that $f(1)=0$ or $f(1)=3$.

Sub-case 1. Consider $f(2) = 0$, $f(1) = 0$. Substituting $f(1) = 0$ in (3), we obtain

$$f(x + xz) = 2f(xz) + f(x).$$

This implies that

$$f(1 + z) = 2f(z), \quad f(2x) = 3f(x).$$

Using these relations, we may obtain

$$f(1 + 2z) = 2f(2z) = 6f(z).$$

This gives

$$f(2 + 2z) = f(1 + 1 + 2z) = 2f(1 + 2z) = 12f(z).$$

But we also have

$$f(2 + 2z) = f(2(1 + z)) = 3f(1 + z) = 6f(z).$$

Comparing these expressions, we conclude that $f(x) \equiv 0$. Note that this is another constant solution for (1).

Sub-case 2. Suppose $f(2) = 0$ and $f(1) = 3$. Taking $x = 1$ and $f(1) = 3$ in (3), we get

$$f(1 + z) = -f(z) + 3.$$

It follows that

$$f(2 + z) = -f(1 + z) + 3 = f(z).$$

Thus f is periodic with period 2. Taking $x = 2$ in (3), we see that

$$f(2 + 2z) = -f(2z).$$

The periodicity of f shows that $f(2 + 2z) = f(2z)$. Thus $f(2z) = 0$ for all z and hence $f(x) \equiv 0$. But this contradicts $f(1) = 3$.

Case 2. Now consider the possibility $f(2) = 2$. Taking $z = 1$ in (3), we get

$$f(2) = \big(3 - f(1)\big)f(1).$$

This gives $f(1)^2 - 3f(1) + 2 = 0$. Thus $f(1) = 2$ or $f(1) = 1$.

Sub-case 1. Let $f(2) = 2$ and $f(1) = 2$. Taking $x = 1$ in (3), we get

$$f\big(1 + z\big) = f(1) = 2.$$

This implies that $f(x) = 2$ for all real x. But this contradicts $f(0) = 0$.

Sub-case 2. Suppose $f(2) = 2$ and $f(1) = 1$. Now (3) reduces to

$$f\big(x + xz\big) = f\big(xz\big) + f\big(x\big),$$

If we set $xz = y$, then we see that for $x \neq 0$, y varies over $\mathbb{R}$ as z varies over $\mathbb{R}$. Hence

$$f(x + y) = f(x) + f(y),$$

for all $x \neq 0$ and y. Since $f(0) = 0$, this is also valid for $x = 0$. Hence (1) shows that

$$f(xy) = f(x)f(y),$$

for all x, y. Thus f is both additive and multiplicative. Since $f(1) = 1$, it follows that $f(x) = x$ for all real x.

6.3 If $n = 1$, then $P(x) = x + a_0$. Thus $P(x) = 0$ has root $-a_0$ which can be equal to a_0 only if $a_0 = 0$. Hence $n > 1$. Write

$$P(x) = (x - a_0)(x - a_1)(x - a_2) \cdots (x - a_{n-1}).$$

Thus $a_0 = (-1)^n a_0 a_1 a_2 \cdots a_{n-1}$. We therefore obtain

$$(-1)^n a_1 a_2 \cdots a_{n-1} = 1.$$

In particular $|a_j| = 1$ for $1 \le j \le n-1$. We consider two cases: $|a_0| = 1$ and $|a_0| > 1$.

Suppose $|a_j| = 1$. Note that $a_j = \pm 1$ for $0 \le j \le n-1$. Hence we write

$$P(x) = (x-1)^{\alpha}(x+1)^{\beta},$$

where $\alpha + \beta = n (> 1)$. Expanding this and comparing the coefficients of x^{n-1} and x^{n-2}, we obtain

$$\beta - \alpha = a_{n-1} = \pm 1, \quad \binom{\beta}{2} + \binom{\alpha}{2} - \alpha\beta = a_{n-2} = \pm 1.$$

An easy computation gives $-(\alpha+\beta) = \pm 2 - 1$. Thus $\alpha+\beta = 3$. The possibilities are $(\alpha, \beta) = (1,2)$ or $(2,1)$. Thus the polynomials are

$$P(x) = x^3 + x^2 - x - 1, \quad P(x) = x^3 - x^2 - x + 1.$$

We see that $P(x) = x^3 + x^2 - x - 1$ is the only one which is admissible.

Suppose $|a_0| \ge 2$. We see that

$$\begin{aligned} 0 = P(a_0) &= \left| a_0^n + a_{n-1}a_0^{n-1} + \cdots + a_1 a_0 + a_0 \right| \\ &\ge |a_0^n - |a_0|^{n-1} - \cdots - |a_0|^2 - 2|a_0| \\ &= \frac{|a_0|\big(|a_0| - 2\big)\big(|a_0|^{n-1} - 1\big)}{|a_0| - 1} \ge 0. \end{aligned}$$

Thus $|a_0| = 2$, and equality holds in the inequality. This implies that a_0 is negative and hence $a_0 = -2$. Moreover the signs of $a_{n-j}a_0^{n-j}$ for $1 \le j \le n-1$ are the same as that of a_0, so that

$$a_{n-j} = (-1)^{n-j+1}, \quad \text{for } 1 \le j \le n-1.$$

If $n > 2$, then $a_2 = (-1)^3 = -1$ and

$$0 = f(a_2) = f(-1)$$
$$= (-1)^n + \sum_{j=1}^{n-1} (-1)^{n-j+1}(-1)^{n-j} - 2 = -n.$$

Thus $n = 2$ and $P(x) = x^2 + x - 2$. We see that this also has the required property.

6.4 Changing x to $f(x)$, we get

$$\begin{aligned} f\big(f(x)f(y)\big) &= f\big(f(x)y\big) + f(x) \\ &= f(xy) + y + f(x). \end{aligned}$$

Interchanging x and y, we also get

$$f\big(f(x)f(y)\big) = f(xy) + x + f(y).$$

Thus it follows that $f(x) - x = f(y) - y$ for all $x, y \in \mathbb{R}^+$. Hence $f(x) = x + c$ for some constant c. Putting this in the given relation, we obtain $xy + cx + c = xy + c + x$, for all $x \in \mathbb{R}^+$. We conclude that $c = 1$ and hence $f(x) = x + 1$.

6.5 Putting $x = 1$, we get a quadratic equation in $P(1)$ and hence $P(1) = 1 \pm \sqrt{3}$. If $P(1) = 1 + \sqrt{3}$, we may write $P(x) = (x-1)Q(x) + 1 + \sqrt{3}$. This gives a relation for $Q(x)$:

$$4(x+1)Q(2x^2 - 1) = (x-1)Q(x)^2 + 2(1+\sqrt{3})Q(x).$$

Taking $x = 1$, we see that $Q(1) = 0$. Thus $(x-1)$ divides $Q(x)$. We may use the induction to prove that $(x-1)^n$ divides $Q(x)$ for all positive integers n. Thus $Q(x) \equiv 0$ and $P(x) = 1 + \sqrt{3}$ for all x. Similarly, $P(1) = 1 - \sqrt{3}$ gives $P(x) = 1 - \sqrt{3}$ for all x.

6.6 Suppose such a function exists. Write the given relation in the form

$$f(x) - f(x+y) \geq \frac{f(x)y}{f(x)+y}.$$

Thus f is a strictly decreasing function. Fix $x > 0$ and choose a positive integer such that $nf(x+1) \geq 1$. For $0 \leq k \leq n-1$, show that

$$f\left(x + \frac{k}{n}\right) - f\left(x + \frac{k+1}{n}\right) \geq \frac{1}{2n},$$

and hence $f(x) - f(x+1) \geq 1/2$. This holds for any $x > 0$. Fix some $x > 0$ and choose a positive integer $m \geq 2f(x)$. Telescope $f(x) - f(x+m)$ and show that

$$f(x) - f(x+m) \geq \frac{m}{2} \geq f(x).$$

This implies $f(x+m) < 0$.

6.7 Define a function $h : \mathbb{R} \to \mathbb{R}$ by

$$f\left(e^{2\pi it}\right) = e^{2\pi i h(t)}.$$

Then h is continuous on $\mathbb{R}$. We observe that

$$e^{2\pi i h(t+1)} = f\left(e^{2\pi i(t+1)}\right) = f\left(e^{2\pi it}\right) = e^{2\pi i h(t)}.$$

Thus $h(t+1) - h(t) = n$ is an integer. Using $f(z^2) = f(z)^2$, we can show that $h(2t) - 2h(t) = m$ is also an integer. Consider $g(t) = h(t) - nt + m$. It is easy to check that $g(t+1) = g(t)$, and $g(2t) = 2g(t)$ for all t. The only such function is *zero* function. Suppose $g(a) = b \neq 0$ for some a. The induction gives $g(2^k a) = 2^k b$ for all natural numbers k. Hence g is unbounded. But the continuity of g and $g(t+1) = g(t)$ imply that g is bounded Hence $g(t) = 0$ for all t. This gives $h(t) = nt - m$ for all t. Thus

$$f\left(e^{2\pi it}\right) = e^{2\pi i(nt-m)} = (-1)^m \left(e^{2\pi it}\right)^n.$$

This shows that $f(z) = (-1)^m z^n$ for all z on the unit circle. But $f(z^2) = f(z)^2$ implies that m is even and hence $f(z) = z^n$ for all z on the unit circle.

6.8 Put $f(0) = s$. Then $f(f(y)) = s^2 + y$ for all y. This shows that f is a bijection. Let t be such that $f(t) = 0$. Taking $x = t$, conclude that $s = 0$. Thus $f(x) = 0$ if and only if $x = 0$ and hence $t = 0$. Put $y = 0$ in the given relation and get $f(xf(x)) = f(x)^2$. Changing x to $f(x)$, conclude that $f(x)^2 = x^2$ for all x. Thus $f(x) = \pm x$. Prove that $f(x) = x$ for some x and $f(y) = -y$ for some $y \neq x$ are not compatible. Hence $f(x) = x$ for all x or $f(x) = -x$ for all x.

6.9 Show that $f(1) = f(0)$. Taking $x = y = 1$, conclude that $f(2) = f(1) = f(0)$. Show that $f(m^2) = f(0)$ for all $m \in \mathbb{N}_0$. Thus $f(4) = f(9) = f(0)$. Now $x = 3$, $y = 4$ gives $3f(4) + 4f(3) = 7f(5^2) = 7f(0)$. Hence $f(3) = f(0)$. Similarly $x = 1$, $y = 2$ gives $f(5) = f(0)$. Using $x = y = 2$, get $f(8) = f(0)$. Using $x = 8$, $y = 6$, show that $f(6) = f(0)$. Taking $x = 7$ and $y = 1$, we get $f(7) = f(0)$. Put $x = 3$ and $y = 1$ to get $f(10) = f(0)$. This shows that $f(m) = f(0)$ for $0 \leq m \leq 10$. Use suitable identities and the induction to prove that

$$f(5k+1) = f(5k+2) = f(5k+3) \\ = f(5k+4) = f(5k+5) = f(0).$$

Move in steps of *five*.

6.10 First show that $f(f(y) = y$ and $f(x^2 + y) = xf(x) + f(y)$. Taking $y = 0$, get $f(x^2) = xf(x) + f(0)$. Replacing x by $x + y$, prove that $xf(x) + yf(y) + f(2xy) = (x+y)f(x+y) + f(0)$. Now replace x by x^2 and get

$$x^2 f(0) + f(2x^2 y) = x^2 f(y) + xyf(x) + f(0).$$

Putting $y = 1/2$, prove that $xf(x) = 2x^2(f(1/2) - f(0))$. Now $x = 1/2$ gives $f(0) = 0$ and hence $xf(x) = x^2 f(1/2)$. This shows that $f(x) = cx$ for all x, where c is some constant. Prove that $c = \pm 1$ using the original relation. We get two solutions: $f(x) = x$; $f(x) = -x$.

6.11 Show that $f(xf(x)) = x^2 + s$, where $s = f(0)$. Prove that $f(f(1) + y) = 1 + f(y)$ for all y. Multiplying this by $y + f(1)$ and repeatedly using the given relation, prove that $f(1) = \pm 1$.

If $f(1) = 1$, using $f(1 + y) = 1 + f(y)$ prove that $s = 0$.This gives $f(y^2) + y^2 = 2yf(y)$ and $f(xf(x)) = x^2$. Use these effectively to prove that $f(y + f(y)) = 2y$ and $f(2y + y^2) + y^2 = 2f(y) + 2yf(y)$. These imply $f(2y^2) = 2f(y^2)$. Putting all these together, we get $(y - f(y))^2 = 0$. Hence $f(y) = y$ for all y.

If $f(1) = -1$, again prove that $f(0) = 0$, show that $g(x) = -f(x)$ satisfies $g(xg(x) + y) = x^2 + g(y)$. Since $g(1) = -f(1) = 1$, it follows that $g(y) = y$ and hence $f(x) = -x$ for all x.

6.12 Easy! The only solution is $f(x) = x$.

6.13 The only solutions are $f(x) = x$ and $f(x) = -x$.

6.14 We show that $f(x) = 1/x$ is the only solution. We do this in several steps.

We first show that $f(x)$ is a non-increasing function. Suppose the contrary; say $0 < a < b$ implies that $f(a) < f(b)$, for some a, b. Then

$$w = \frac{bf(b) - af(a)}{b - a}$$

is in $(0, \infty)$. It is easy to check that $w > f(b)$ using $f(b) > f(a)$. Thus $w > f(b) > f(a)$. Taking $x = a$ and $y =$

$w - f(a)$, we get

$$f(w) = af\left(1 + ab\frac{(f(b) - f(a))}{b - a}\right)$$

Similarly, $x = b$ and $y = w - f(b)$ gives

$$f(w) = baf\left(1 + ab\frac{(f(b) - f(a))}{b - a}\right)$$

Thus $a = b$ contradicting $a < b$. We conclude that $a < b$ implies that $f(b) \leq f(a)$.

Taking $x = y = 1$, we get $f\big(f(1) + 1\big) = f(2)$. Similarly, putting $x = 1$, $y = 2$, we obtain $f\big(f(1) + 2\big) = f(3)$; $x = 2$, $y = 1$ implies $f\big(f(2) + 1\big) = 2f(3)$. Thus

$$\begin{aligned} 2f(3) = f\big(f(2) + 1\big) &= f\Big(f\big(f(1) + 1\big) + 1\Big) \\ &= \big(f(1) + 1\big)f\Big(1 + \big(f(1) + 1\big)\Big) \\ &= \big(f(1) + 1\big)f\big(f(1) + 2\big) \\ &= \big(f(1) + 1\big)f(3). \end{aligned}$$

This gives $f(1) + 1 = 2$ and hence $f(1) = 1$.

Suppose $x > 1$, and put $y = 1 - (1/x)$. Then

$$f\left(f(x) - \frac{1}{x} + 1\right) = xf\big(1 + x - 1\big) = xf(x).$$

If $f(x) > (1/x)$, we see that $f(x) - (1/x) + 1 > 1$ and the monotonicity of $f(x)$ gives

$$f\left(f(x) - \frac{1}{x} + 1\right) \leq f(1) = 1.$$

Thus $xf(x) \leq 1$, contradicting $f(x) > (1/x)$.

If $f(x) < (1/x)$, we see that $f(x) - (1/x) + 1 < 1$, and

$$f\left(f(x) - \frac{1}{x} + 1\right) \geq f(1) = 1,$$

and hence

$$1 \leq f\left(f(x) - \frac{1}{x} + 1\right) = xf(x) < 1,$$

which again is impossible. We conclude that $f(x) = \dfrac{1}{x}$, for all $x > 1$.

Now, take any $x > 0$. Then $f(x) + 1 > 1$, so that

$$f\big(f(x) + 1\big) = \frac{1}{f(x) + 1}.$$

Putting $y = 1$ in the given relation, we get

$$f\big(f(x) + 1\big) = xf(1 + x) = \frac{x}{1 + x};$$

we have used $1 + x > 1$ in the last equality. Thus we obtain

$$\frac{1}{f(x) + 1} = \frac{x}{1 + x},$$

for all $x > 0$. This gives $f(x) = 1/x$ for all $x > 0$.

6.15 No. Show that f is strictly increasing on $\mathbb{N} \setminus \{1\}$. Hence $f(n) \geq n - 1$ for all $n \geq 2$. If $f(n) = n - 1$ for some n, prove that $n = 4$. Hence there exists some $n_0 \geq 2$ such that $f(n) \geq n$ for all $n \geq n_0$. Repeat the same argument twice to conclude that there exists $n_2 \geq 2$ such that $f(n) \geq n + 2$ for all $n \geq n_2$. Taking $m = f(n_2)$, and using the given relation repeatedly, prove that

$$-f(n_2 + 1) = f\big(f(n_2 + 1)\big) + \cdots + f\big(f(m - 2)\big).$$

This contradiction proves the non-existence of such an f.

6.16 Consider the equation

$$f(x+yf(x)) = f(x) + xf(y). \tag{1}$$

Taking $x = 1$ and $y = 0$ in (1), we get $f(0) = 0$. Suppose $f(x) \not\equiv 0$. If $f(x) = 0$ for some x, then $0 = f(x) = f\big(x+yf(x)\big) = f(x)+xf(y) = xf(y)$. Choosing y such that $f(y) \neq 0$, we see that $x = 0$. Thus $f(x) = 0$ implies $x = 0$.

Putting $x = 1$, we get $f\big(1 + yf(1)\big) = f(1) + f(y)$, for all $y \in \mathbb{R}$. If $f(1) \neq 1$, we may choose $y = 1/\big(1 - f(1)\big)$. This gives $1 + yf(1) = y$ and hence we obtain $f(y) = f\big(1 + yf(1)\big) = f(1) + f(y)$ forcing $f(1) = 0$. This leads to the absurdity that $1 = 0$. Hence $f(1) = 1$. Taking $x = 1$ in (1), we obtain $f(1 + y+ = 1 + f(y)$ for all $y \in \mathbb{R}$.

Take any $x \neq 0$ so that $f(x) \neq 0$. Choosing $y = 1/f(x)$ in (1), we obtain

$$f(x+1) = f(x) + xf\left(\frac{1}{f(x)}\right).$$

We conclude that $f\left(\dfrac{1}{f(x)}\right) = \dfrac{1}{x}$ for all $x \neq 0$. Replacing y in (1) by $y/f(x)$ with $x \neq 0$, we get

$$f(x+y) = f(x) + xf\left(\frac{y}{f(x)}\right), \forall x \neq 0, y \in \mathbb{R}. \tag{2}$$

Replacing x by $1/f(x)$ in (2), this changes to

$$f\left(\frac{1+yf(x)}{f(x)}\right) = \frac{1}{x} + \frac{1}{f(x)} f(yx), \tag{3}$$

valid for all $x \neq 0$ and $y \in \mathbb{R}$. Replacing y in (2) by $1+yf(x)$, we also obtain

$$f\big(x+1+yf(x)\big) = f(x) + xf\left(\frac{1+yf(x)}{f(x)}\right).$$

In view of (3), $f(x+1) = f(x)+1$ and (1), this simplifies to $xf(y)f(x) = xf(yx)$. Since $x \neq 0$, we get $f(xy) = f(x)f(y)$. Though it is valid a-priori for $x \neq 0$ and all $y \in R$, we see that $f(xy) = f(x)(f(y)$ for all $x, y \in \mathbb{R}$ because of $f(0) = 0$. Using this in (2), we also get the additivity:

$$f(x+y) = f(x) + xf(y)f\big(1/f(x)\big) = f(x) + f(y),$$

which may be seen to be valid for all $x, y \in \mathbb{R}$. Thus f is both additive and multiplicative. Since f is not identically zero function, it follows that $f(x) = x$ for all $x \in \mathbb{R}$.

6.17 Consider $g(x) = f(x) - f(0)$. Show that g also satisfy the same functional equation and $g(0) = 0$. It implies that $g(x^2) = xg(x)$. Use this to conclude that $g(x)/x = g(y)/y$ for all $x, y \neq 0$. Hence $g(x) = kx$ and $f(x) = kx + c$.

6.18 First show that f is surjective. Let s be such that $f(s) = 0$. Prove that $f(y) - s = s + f\big(f(y) - s\big)$ for all y. Now the surjectivity of f implies that $f(x) = x - s$.

6.19 Write $P(x) = \sum_{j=0}^{n} a_j x^j$. Consider $(a, b, c) = (6x, 3x, -2x)$. Show that for such a triple we get $P(3x) + P(5x) + P(-8x) = 2P(7x)$. Obtain relations involving coefficients. Show that $a_j = 0$ for $j \neq 2, 4$.

6.20 Show that $f(1) = 1$. Observe that $m^2 + f(n) \leq f(m)^2 + n$ for all m, n. Using this, conclude that $f(m) \geq m$ and $f(n) \leq n$. Hence $f(n) = n$ for all $n \in \mathbb{N}$.

6.21 Note that $f(x) \equiv 0$ is a solution. Replacing once y by $f(x)$ and again y by $2f(y) - f(x)$, show that

$$f\big(2f(x) - 2f(y)\big) = f(0) + 4\big(f(x) - f(y)\big)^2.$$

Choose x_0 such that $f(x_0) \neq 0$. Taking $y = x/8f(x_0)$, obtain a relation for $f(x)$ and use this to conclude that $f(x) = f(0) + x^2$. Thus $f(x) = x^2 + c$ gives another class of solutions.

6.22 First show that if α is a root of $P(x) = 0$, then $\alpha = 0$ or $|\alpha| = 1$. Take $P(x) = x^m(x-1)^n Q(x)$, where $Q(x) = 0$ has no roots at $x = 0$ or 1. Show that $m = 0$ and Q satisfies

$$Q(x^2) = Q(x)Q(x+2).$$

Using this conclude that $Q(x)$ is a constant polynomial. Show that this constant is 0 or 1. Thus th class of polynomials is $P(x) \equiv 0$ and $P(x) = (x-1)^n$

6.23 Let $P(x) = ax^3+bx^2+cx+d$, where $a \neq 0$ and a, b, c, d are real numbers. Observe that $P(x+y) \geq P(x) + P(y)$ is equivalent to

$$xy\big(3a(x+y) + 2b\big) \geq d.$$

Conclude that $a > 0$ and $d \leq 0$. Taking $-d = u$, we have $u \geq 0$ and the equivalent condition is

$$3axy(x+y) + u \geq -2bxy,$$

for all $x, y \geq 0$. For positive reals x, y, this may be written as

$$3ax + 3ay + \frac{u}{xy} \geq -2b.$$

Now the AM-GM inequality shows that

$$3ax + 3ay + \frac{u}{xy} \geq 3\big(9a^2\big)^{1/3}.$$

Since equality can hold in the AM-GM inequality, a necessary and sufficient condition for $P(x+y) \geq P(x) + P(y)$ is

$$d \leq \frac{8b^3}{243a^2}.$$

6.24 Taking $f(1) = u$, we observe $u > 0$ and $u = f(u)$. Taking $y = 1$ and $x = u$, we see that u satisfies the cubic $2u^3 = u(u+1)$. This force $u = 1$. Taking $x = 1$, we get $1 + f(y) = (1+y)f(y)$. Hence $f(y) = 1/y$ for all $y > 0$.

6.25 Take $Q(x) = P(x) - x^2$. Then $Q(x) - Q(x+1) = Q(x-1) - Q(x)$. This implies that $Q(x) = ax + b$ for some constants a, b. Thus $P(x) = x^2 + ax + b$.

6.26 Observe $f(1) = 1$. Hence $f(n) + 1$ divides $(n+1)^2$ and $f^2(m) + 1$ divides $(m^2+1)^2$. Taking $n = p - 1$, where p is a prime, conclude that $f(p-1) + 1 = p$ or p^2. Show that $f(p-1) + 1 = p^2$ is not possible using the second divisibility above. Thus $f(p-1) = p - 1$ for all primes. Show that for any k with $f(k) = k$, the number $k^2 + f(n)$ divides $(k^2+n)^2$. Use this to conclude that $k^2 + f(n)$ divides $(n - f(n)^2$. It follows that $f(n) = n$ for all n.

6.27 Observe $f(n) \geq n$. Consider $F(n) = f(n) - n$. Show that F satisfies

$$F\big(F(n) + m\big) = F(m) + n.$$

Using this, conclude that $F(1) = 1$ and $F(n+1) = F(n) + F(1)$ for all $n \geq 1$. Thus $F(n) = nF(1)$. It follows that $F(n) = n$ and $f(n) = 2n$.

6.28 Show that f is one-one. Choose positive real x such that $x + 1 = x^2$. Use this x to prove that $f(1) = 0$, which is impossible.

6.29 Observe that $f(x) \equiv 0$ is a solution. Assume $f(x) \not\equiv 0$ and choose a such that $f(a) \neq 0$. Changing y to $-f(x)$, prove that $f(-f(x)) = c + f(x)^2$ where $c = f(0)$. Changing y to $-f(y)$, conclude that

$$f\big(f(x) - f(y)\big) = \big(f(x) - f(y)\big) + c^2.$$

Putting $x = a$, prove that every real z can be expressed in the form $f(u) - f(v)$ for some u and v. Conclude that $f(z) = z^2 + c$ for all real z.

6.30 Taking $x = y = t > 0$, we get

$$4f(t) \le f(2t), \quad f(2t) \le 4f(t).$$

Thus $f(2t) = 4f(t)$ for all $t > 0$. The induction gives $f(2^m t) = 2^{2m} f(t)$ for all $t > 0$ and $m \in \mathbb{N}$. Take $g(x) = f(x)/x$ for $x > 0$ and $g(0) = 0$. We show that $g(nt) = ng(t)$ for all $n \in \mathbb{N}$ and $t > 0$. This is true for $n = 2^m$. Using $g(x + y) \le g(x) + g(y)$, it is immediate that $g(nx) \le ng(x)$ for all $x > 0$. For any n, choose m such that $2^{m-1} \le n < 2^m$. Observe

$$\begin{aligned} 2^m g(t) = g(2^m t) &\le g(nt) + g\big((2^m - n)t\big) \\ &\le ng(t) + (2^m - n)g(t) = 2^m g(t). \end{aligned}$$

Thus equality holds and $g(nt) = ng(t)$ for all n.

Next we show that g is monotonically decreasing. Observe

$$f(t) + f(2t) \le \frac{f(3t)}{2},$$

and hence

$$g(t) + 2g(2t) \le \frac{3}{2} g(3t).$$

This leads to $10g(t) \le 9g(t)$ and hence $g(t) \le 0$ for all $t \ge 0$. For any x, y with $0 < x \le y$, we thus have

$$g(x) \ge g(x) + g(y - x) \ge g(y).$$

Let $g(1) = a \le 0$. We show that $g(t) = at$ for all $t > 0$. Suppose $g(t) < at$ for some $t > 0$. Choose a positive rational number p/q such that $t < (p/q)$ and $g(t) < (pa/q)$. Using $g(nt) = ng(t)$ for all n, we can show that $g(p/q) = (p/q)g(1)$. Thus $g(t) \ge g(p/q) = (pa/q)$, a contradiction. Similarly, $g(t) > at$ for some $t > 0$ also fails. Hence $g(t) = at$ for all $t > 0$. This gives $f(x) = ax^2$ for all $x > 0$, where $a \le 0$. It is easy to check that both the inequalities are satisfied by this function.

6.31 Distinguish $k = 0$ and $k \neq 0$. Ans:
for $k \neq 0$:

$$f(x,y) = \begin{cases} x^k g(y/x), & \text{for } x \neq 0, \\ cy^k, & \text{for } x = 0,\ y \neq 0, \\ 0, & \text{for } x = y = 0; \end{cases}$$

for $k = 0$:

$$f(x,y) = \begin{cases} g(y/x), & \text{for } x \neq 0, \\ c, & \text{for } x = 0,\ y \neq 0, \\ \text{arbitrary}, & \text{for } x = y = 0; \end{cases}$$

here g is an arbitrary function.

6.32 We have

$$f(x,y) = f(0+x, (y-x)+x) = f(0, y-x) + x,$$

and

$$f(0, y-x) = f\big(0\cdot(y-x), 1\cdot(y-x)\big) = (y-x)f(0,1), \text{for } y \neq x.$$

Taking $f(0,1) = \lambda$, we have

$$f(x,y) = \lambda y + (1-\lambda)x.$$

This also holds for $x = y$ from the given relation.

6.33 Take $x = 0$ to get $f(y) = f(0)g(y) + h(y)$. Eliminate $h(y)$ from this and the given relation to get

$$\varphi(x+y) = \varphi(x)g(y) + \varphi(y), \qquad (\star)$$

where $\varphi(t) = f(t) - f(0)$; and $\varphi(0) = 0$. If $g(t) \equiv 1$, then φ is a continuous additive function. Hence $\varphi(x) = cx$ and we get

$$f(x) = cx + f(0), g(x) \equiv 1, h(x) = cx.$$

If $g(t) \not\equiv 1$, we interchange the roles of x, y in $(\star)$, and compare the two expressions for $\varphi(x+y)$ to get

$$\varphi(x)\big(g(y)-1\big) = \varphi(y)\big(g(x)-1\big).$$

Choosing y_0 such that $g(y_0) \neq 1$, we solve for $\varphi(x)$:

$$\varphi(x) = \frac{\varphi(y_0)}{g(y_0)-1}\big(g(x)-1\big) = \lambda\big(g(x)-1\big),$$

where λ is a constant. Here again consider $\lambda = 0$ and $\lambda \neq 0$. If $\lambda = 0$, then

$$f(x) = f(0) = \alpha, \quad g(x) \text{ arbitrary}, \quad h(x) = \alpha\big(1-g(x)\big).$$

If $\lambda \neq 0$, substitute $\varphi(x)$ in $(\star)$. We see that g satisfies $g(x+y) = g(x)g(y)$, and hence either $g(x) \equiv 0$ or $g(x) = c^x$ for some positive real number c. In this case we get

$$f(x) = \alpha - \lambda, \quad g(x) = 0, \quad h(x) = \alpha - \lambda;$$

and

$$f(x) = \lambda c^x + \alpha - \lambda; \quad g(x) = c^x; \quad h(x) = (\alpha - \lambda)(1 - c^x)$$

where α and $\lambda \neq 0$ are some constants.

6.34 Show that $P(1) = 0$ or $P(0) = 1$. If $P(1) = 0$, prove that $P\big(2^{2^n}\big) = 0$ for all natural numbers n. This implies that $P(0) = 1$ and hence 0 cannot be a root of $P(x) = 0$. Use this to prove that any root α of $P(x) = 0$ must lie on the unit circle. If $P(\alpha) = 0$, then $P(\alpha+1) = 0$ and hence $|\alpha+1| = 1$. These two together imply that

$$\alpha = \frac{-1}{2} \pm i\frac{\sqrt{3}}{2}.$$

Show that $(x-\alpha)(x-\overline{\alpha})$ is a factor of $P(x)$. This forces $P(x) = x^2 + x + 1)^n$ for some $n > 1$.

6.35 The function $g(x) = f(x) - xf(1) + xf(0) - f(0)$ also has the same property as that of f. Moreover $g(0) = g(1) = 0$. Now g attains its maximum M and minimum m on $[0,1]$. Consider the sets

$$E_M = \{x \in [0,1] : g(x) = M\}, \quad E_m = \{x \in [0,1] : g(x) = m\}.$$

Both are non-empty *closed* sets. Let $y = \max E_M$. Then $y \in E_M$. If $0 < y < 1$, then we can find $h > 0$ such that $0 \le y - h < y + h \le 1$ and

$$2M = 2g(y) = g(y-h) + g(y+h) \le M + M = 2M.$$

This forces $g(y-h) = g(y+h) = M$. But then $y + h \in E_M$, which is impossible. Thus $y = 0$ or $y = 1$. Since $g(0) = g(1) = 0$, we get $M = 0$ or $g(y) \le 0$ for all $y \in [0,1]$. Similarly $g(y) \ge 0$ by considering the set E_m. This gives $g(x) \equiv 0$ and hence $f(x) = ax + b$.

6.36 Observe $f(x)$ is also onto. Hence f is a continuous bijection on $\mathbb{R}$. This implies that f is a strictly monotone function. We show that f is strictly increasing. Suppose the contrary. Then f must be strictly decreasing. Thus $x < y$ implies $f(y) < f(x)$. Thus $2x - f(x) < 2y - f(y)$ and hence

$$y = f\big(2y - f(y)\big) < f\big(2x - f(x)\big) = x.$$

This contradicts $x < y$. Thus f is a strictly increasing function and

$$f(x) + f^{-1}(x) = 2x,$$

for all $x \in \mathbb{R}$. This implies that $f(x) = x + d$ for some constant d. (See problem 5.16.) Since $f(x_0) = x_0$, it follows that $d = 0$. Thus $f(x) = x$ for all x.

6.37 Put $x = y = 0$ to see that $f(0) = 0$ or $f(0) = \pm 1$. If $f(0) = 0$, we have $f(x) \equiv 0$. If $f(x)$ is a solution, so is $-f(x)$. Thus all we need to consider is $f(0) = 1$. In this case $f(y)f(-y) = f(y)^2$. This shows that $f(y)^2 = f(-y)^2$. If $f(-y) = -f(y)$ prove that $f(y) = 0$. This shows that it suffices to consider the case $f(y) = f(-y)$. Put $y = x$ to get $f(2x) = f(x)^4$. Use this to prove that whenever $f(x_0) = 0$ for some x_0, we get $f(0) = 0$ contradicting $f(0) = 1$. Thus $f(x) \neq 0$ for all x. Since $f(x) = f(x/2)^4$, it follows that $f(x) > 0$ for all x. Use induction to prove that $f(nx) = f(x)^{n^2}$ for all natural numbers n. From this prove that $f(r) = f(1)^{r^2}$ for all rational numbers r. The continuity forces $f(x) = f(1)^{x^2}$ for all x.

6.38 Taking $x = 0$ in the given equation, we get

$$f\big(f(y)\big) = y\big(1 - f(0)\big) + 2f(0).$$

This shows that f is one-one and onto. Let s be such that $f(s) = 0$. Take $y = s$ to get $f(0) = s\big(f(x) - 1\big)$. Since $f(x)$ is not a constant function, it follows that $s = 0$. Since f is one-one, $f(x) \neq 0$ for any $x \neq 0$. Moreover $f\big(f(y)\big) = y$ for all y. Replacing y by $f(y)$, we get

$$f(xy) + f(x) + f(y) = f(x + y) + f(x)f(y), \qquad (1)$$

for all x, y. Taking $x = y = 2$, we see that $f(2)^2 = 2f(2)$. Thus $f(2) = 0$ or $f(2) = 2$. Using injectivity of f and $f(0) = 0$, we get $f(2) = 2$. If $f(1) = a$, taking $x = y = 1$ in (1), we get $a^2 + 2 = 3a$. Hence $a = 2$ or 1. But $f(2) = 2$ forces $a = f(1) = 1$. Taking $y = 1$ in (1), we get $f(x + 1) = f(x) + 1$. Changing x by $x + 1$ in (1), we obtain

$$f(xy + y) + f(x) + f(y) = f(x)f(y) + f(y) + f(x + y). \qquad (2)$$

Comparing (1) and (2), we see that

$$f(xy + y) = f(xy) + f(y),$$

for all x, y. If a, b are non-zero, we can find x, y such that $a = xy$ and $b = y$. Thus we see that $f(a+b) = f(a) + f(b)$ for all non-zero reals a, b. It obviously holds for other values as well. Hence f is an additive function on $\mathbb{R}$. This with (1) imply that f is also multiplicative. Hence $f(x) \equiv 0$ or $f(x) = x$. (For a different solution of (1) with no injectivity of f, see problem 6.2.)

6.39 First of all f is a one-one function. Suppose $f(a) = f(b)$ and $a \neq b$. Then $f(a) + f(n)$ is a common divisor of $(a+n)^k$ and $(b+n)^k$ for all n. Note that $\gcd(a+n, b+n) = \gcd(a+n, b-a)$ and choosing n such that $a+n$ is a prime greater than b, we get $\gcd\left((a+n)^k, (b+n)^k\right) = 1$. Hence $f(a) + f(n) = 1$ for this value of n, which is impossible as $f(a) + f(n) > 1$. Thus f is one-one.

Again $f(b)+f(n)$ divides $(b+n)^k$ and $f(b+1)+f(n)$ divides $(b+n+1)^k$. But note that $\gcd\left((b+n)^k, (b+n+1)^k\right) = 1$. This implies that

$$\begin{aligned}\gcd\big(f(b) + f(n), f(b+1) - f(b)\big) \\ = \gcd\big(f(b) + f(n), f(b+1) + f(n)\big) = 1.\end{aligned}$$

This implies $f(b+1) - f(b) = \pm 1$. If not, choose a prime p that divides $f(b+1) - f(b)$. Choose $a \in \mathbb{N}$ such that $p^a > b$ and take $n = p^a - b$. Use this to prove that p divides $f(n) + f(b)$. But then $\gcd\big(f(b) + f(n), f(b+1) - f(b)\big) \geq p$, a contradiction. This proves $f(b+1) - f(b) = \pm 1$. Since f is injective, either $f(b+1) - f(b) = 1$ for all b or $f(b+1) - f(b) = -1$ for all b. If $f(b+1) - f(b) = -1$ for all b, we see that $f(n) = f(1) + 1 - n < 0$ for large n. Thus $f(b+1) - f(b) = 1$ for all b. This further gives $f(n) = n + c$ for all n, where $c = f(1) - 1$. If $c > 0$, take a prime $p > 2c$. Then it is easy to see that p divides $f(1) + f(p-1) = p + 2c$. But then p divides $2c$, which is a contradiction. Hence $c = 0$ and $f(n) = n$ for all n.

6.40 We see that $f(0) \geq 0$ by putting $x = y = 0$. Thus $f(0) = 0$. Taking $y = -x$, we get $0 = f(0) = f(x - x) \leq f(x) + f(-x)$ so that $f(x) \geq -f(-x) \geq -(-x) = x$. It follows that $f(x) = x$. Note that $f(x) = |x|$ satisfies $f(x + y) \leq f(x) + f(y)$ and $f(x) \geq x$. But $f(x) \neq x$.

6.41 Let $c = f(1)$. Then $f(c\lambda) = \lambda$ and hence $f(\lambda) = c\lambda$, for all real λ. This gives $c^2\lambda = \lambda$ and thus $c = \pm 1$. Thus $f(x) = x$ or $f(x) = -x$ at each x. Show that for any non-zero λ, μ, it cannot happen $f(\lambda) = \lambda$ and $f(\mu) = -\mu$ simultaneously. In the second case, show that $k = 1$ or -1. Thus $f(x) = \pm x$ or $f(x) = \pm 1/x$.

6.42 Taking $k = f(1)$, we see that $f(n + 1) = k^2 + kf(n)$. Use $3^2 + 3 \cdot 1 = 2^2 + 2 \cdot 4$ to get a polynomial relation for k. Conclude $k = 1$ and hence $f(n+1) = f(n)+1$. By induction $f(n) = n$ for all n.

6.43 First show that $f(0) = 0$ and $f(1) = 1$. Factorise the first equation:

$$\big(f(2n + 1) - f(2n)\big)\big(f(2n + 1) + f(2n)\big) = 6f(n) + 1.$$

Taking $d = f(2n + 1) - f(2n)$, show that $d^2 + 2df(2n) - 1 = 6f(n)$. Use the second condition to prove that $d \leq 3$. Conclude $d = 1$. Thus $f(2n + 1) - f(2n) = 1$ and $f(2n + 1) + f(2n) = 6f(n) + 1$. We get $f(2n) = 3f(n)$ and $f(2n + 1) = 3f(n) + 1$. Use this to prove (by induction) that if $n = \sum a_j 2^j$ in base 2, the value of $f(n)$ is $\sum a_j 3^j$. This shows that $f(n) < 2003$ if and only if $0 \leq n \leq 127$. Ans: 128.

6.44 Use induction to prove that $f(x + n) = f(x) + \big(n/2\big)\big(2x + (n - 1)\big)$ for all $n \in \mathbb{N}$. For $x < 0$, take $n = -[x]$ so that $x + n$ is the fractional part of x. This implies that $f(x) = \big(x^2 + \{x\}^2 - x + \{x\}\big)/2$, where $\{x\} = x - [x]$, the

fractional part of x. Show that this representation is valid even for $x \geq 0$.

6.45 We observe that whenever $f(x)$ is a solution, so is $f(x)+\alpha x+\beta$ for any real numbers α and β. Hence we may assume that $f(0) = 0$ and $f(a) = 0$ for some $a \neq 0$. Taking $y = 0$ and $z = a$ in the equation

$$(y-z)f(x)+(z-x)f(y)+(x-y)f(z) = g(x+y+z), \quad (1)$$

we get

$$f(x) = -x(a-x)g(x+a), \quad (2)$$

for $x \neq 0, a$. Taking $y = 0$ and $z = y$, we also get

$$\frac{f(x)}{x(x-y)} - \frac{f(y)}{y(x-y)} = g(x+y), \quad (3)$$

for all $x, y \neq 0$ and $x \neq y$. Take $\varphi(x) = f(x)/x$, $x \neq 0$. Then (3) reduces to

$$\varphi(x) - \varphi(y) = (x-y)g(x+y), \quad (4)$$

for all $x, y \neq 0$ and $x \neq y$. This is also valid for $x = y$. Taking $y = -x$, we get

$$\varphi(x) - \varphi(-x) = 2xg(0), \quad (5)$$

for all $x \neq 0$. replacing y by $-y$, we obtain

$$\varphi(x) - \varphi(-y) = (x+y)g(x-y), \quad (6)$$

which is valid for all $x, y \neq 0$. Subtract (4) form (6) and use (5) to get

$$(x+y)\big(g(x-y)-g(0)\big) = (x-y)\big(g(x+y)-g(0)\big), \quad (7)$$

for all $x, y \neq 0$. Fix some $u \neq 0$ and choose v such that $u+v \neq 0$, $u-v \neq 0$. Taking $x = (u+v)/2$ and $y = (u-v)/2$, we get

$$u\big(g(v)-g(0)\big) = v\big(g(u)-g(0)\big), \quad (8)$$

for $v \neq u, -u$. This implies that

$$g(v) = \alpha v + \beta, \tag{9}$$

for all $v \neq u, -u$. However, we have taken u as an arbitrary, non-zero real number. By varying u, we conclude that (9) is valid for all $v \in \mathbb{R}$. Putting the expression for g in (3), we see that $f(x) = ax^3 + bx^2 + cx + D$ for some constants a, b, c, d.

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